Set Theory and C^*-algebras: automorphisms of continuous quotients

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Theorem

- *(Rudin) Assume CH. Then there are nontrivial homeomorphisms of $\beta\omega \setminus \omega$. In fact there are $2^{\aleph_1} > \mathfrak{c}$ automorphisms.*
- *(Shelah, Shelah-Steprans, Velickovic) It is consistent that all homeomorphisms of $\beta\omega \setminus \omega$ are trivial. In fact, it follows from OCA + $MA_{\aleph_1}$.*

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Theorem (Rudin,Shelah,Shelah-Steprans,Velickovic)

The automorphisms structure of the C^*-algebra ℓ_∞ / c_0 is independent of ZFC.

Definition

Let X be locally compact and second countable. An homeomorphism ϕ of $\beta X \setminus X$ is trivial if there are compact sets $K_1, K_2 \subseteq X$ and an homeomorphism $f: X \setminus K_1 \to X \setminus K_2$ such that $\phi(x) = \{f(C) \mid C \in x\}$.

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Let X be locally compact, noncompact, and second-countable. Are all automorphisms of the C^*-algebra $C_b(X)/C_0(X)$ "describable"?

	$CH \Rightarrow \exists$ nontrivial	Forcing Axioms $\Rightarrow$ all trivial
$X = \mathbb{N}$	Rudin	Shelah, Shelah-Steprans, Velickovic
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Fix $n \geq 2$. There is a specific space X_n of dimension n such that we don't know whether CH implies the existence of nontrivial homeomorphisms of $\beta X_n \setminus X_n$.

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- If $A = C_0(X)$, $\mathcal{M}(A) = C(\beta X)$, $\mathcal{M}(A)/A \cong C(\beta X \setminus X)$;

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- If $A = C_0(X)$, $\mathcal{M}(A) = C(\beta X)$, $\mathcal{M}(A)/A \cong C(\beta X \setminus X)$;
- If $A = \mathcal{K}(H)$, $\mathcal{M}(A) = \mathcal{B}(H)$. The corona is the Calkin algebra;
- If A_n are C^*-algebras, define

$$B = \bigoplus A_n = \{(a_n) \mid \|a_n\| \to 0\}, \;\; C = \prod A_n = \{a_n \mid \sup \|a_n\| < \infty\}.$$

Then $\mathcal{M}(B) = C$. $\prod A_n / \bigoplus A_n$ is said the **reduced product**.

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Let A be a nonuntail separable C^*-algebra. Can the automorphisms of $\mathcal{M}(A)/A$ be described?

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Conjecture

CH$\Rightarrow$ No. PFA$\Rightarrow$Yes.

Theorem (Phillips-Weaver, Farah)

If $A = \mathcal{K}(H)$, the answer is independent of ZFC.

Definition

Let A_n, B_n be unital C^*-algebras. An isomorphism $\Phi\colon \prod A_n/\bigoplus A_n \to \prod B_n/\bigoplus B_n$ is trivial if there is an almost permutation f and maps $\phi_n\colon A_n \to B_{f(n)}$ making the following commute

$$
\begin{array}{ccc}
\prod A_n & \xrightarrow{\quad \prod \phi_n \quad} & \prod B_n \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\prod A_n/\bigoplus A_n & \xrightarrow{\quad \Phi \quad} & \prod B_n/\bigoplus B_n
\end{array}
$$

Theorem (McKenney-V.)

Assume OCA_∞ and $MA_{\aleph_1}$. Let A_n, B_n be unital separable C^*-algebras, $\Phi\colon \prod A_n / \bigoplus A_n \to \prod B_n / \bigoplus B_n$ an isomorphism, and suppose that (after a certain $m \in \omega$)

- *each A_n is amenable (as a Banach algebra)*
- *no A_n or B_n can be written as $C_n \bigoplus D_n$, where C_n, D_n are unital.*

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Theorem (Farah-Shelah)

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Theorem (Farah-Shelah)

If A_n are unital and separable C^*-algebras, $\prod A_n/\bigoplus A_n$ is countably saturated.

Theorem

Let A_n be separable amenable and unital. Then whether all automorphisms of $\prod A_n/\bigoplus A_n$ are trivial is independent of ZFC.

$M_n = M_n(\mathbb{C}) = \mathcal{B}(\ell^2(n))$ is the C^*-algebra of $n \times n$ complex valued matrices.
Denote by $\mathbb{M}(\{n_i\})$ the reduced product of M_{n_i}.

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For every sequence n_i there is a subsequence n_{k_i} such that if $X, Y \subseteq \omega$ are infinite then

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Theorem (Folklore+Ghasemi)

Assume CH. For every sequence $\{n_i\}$ there are infinite dimensional C^*-algebras A_n such that $\prod A_n / \bigoplus A_n \cong \mathbb{M}(\{n_i\})$ (all such isomorphisms are nontrivial).

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Suppose A_n, B_n are unital and separable and $\phi_n\colon A_n \to B_n$ are maps inducing an isomorphism $\Phi\colon \prod A_n / \bigoplus A_n \to \prod B_n / \bigoplus B_n$.

1. *Can we find isomorphisms $\psi_n\colon A_n \to B_n$ such that $\pi(\prod \psi_n)) = \psi$? (up to a finite set)*
2. *Can we at least say that A_n and B_n must be isomorphic? (up to a finite set)*

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In case one deals with more complicated structures solving problem 1 is proved (McKenney-V.) to be equivalent to a perturbation theory question (in operator algebras) which is open since the '70's.

Theorem (McKenney-V.)

Assume $OCA_\infty + MA_{\aleph_1}$. Suppose that $\phi\colon \prod A_n/\bigoplus A_n \to \prod B_n/\bigoplus B_n$ is an isomorphism and that A_n and B_n cannot be written as $C_n \oplus D_n$, and they are all unital and separable. Suppose also that each A_n is a limit of finite-dimensional algebras, or is abelian. Then there is an almost permutation f such that up to finite sets

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Corollary

Assume $OCA_\infty + MA_{\aleph_1}$. Let X_i, Y_i be metrizable connected compact spaces, $X = \bigsqcup X_i$ and $Y = \bigsqcup Y_i$. Then $\beta X \setminus X \cong \beta Y \setminus Y$ if and only if there is an almost permutation f such that $X_i \cong Y_{f(i)}$ (if $f(i)$ is defined).

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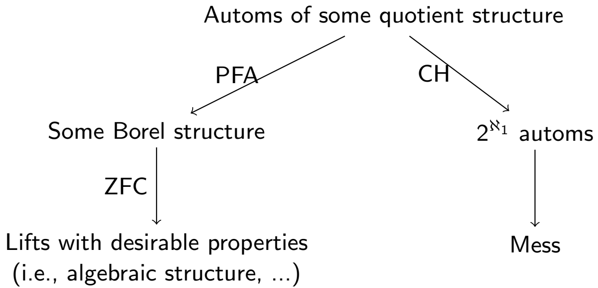

Automs of some quotient structure
PFA
CH
Some Borel structure
$2^{\aleph_1}$ automs
ZFC
Lifts with desirable properties
(i.e., algebraic structure, ...)
Mess

Thank you!