Distributive Aronszajn trees

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When we write "there is a limit $\alpha < \kappa$", we mean "$\exists \alpha \in \mathrm{acc}(\kappa)$".

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Note

If T is a κ-tree, then $\mathbb{P}(T)$ adds a cofinal branch through T. i.e., a sequence $b : \kappa \to H_\kappa$ such that $b \restriction \alpha \in T$ for all $\alpha < \kappa$.

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CTP(κ) asserts that the two hold:

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GCH *is consistent with* CTP($\aleph_1$).

Theorem (Laver-Shelah, 1981)

Assuming a weakly compact, CH *is consistent with* CTP($\aleph_2$).

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For every uncountable cardinal λ, GCH $\implies \neg\,\mathrm{CTP}(\lambda^+)$.

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It is now inevitable to discuss square principles...

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$\square_\lambda$: exists a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for every limit α:

1. C_α is a club in α of order-type $\leq \lambda$;
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$\square_\xi(\kappa, <\mu)$: exists a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that for limit α:

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Square principles and Aronszajn trees are closely related:

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For every uncountable cardinal λ, if GCH $+ \square_{\lambda^+}(\lambda^+, <\lambda^+)$ holds, then there is a λ^+-Aronszajn tree T s.t. $\mathbb{P}(T)$ preserves cardinals.

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Theorem (Ben-David and Shelah, 1986)

For every singular cardinal λ, if GCH $+ \square_{\lambda}(\lambda^+, <\lambda^+)$ *holds, then there is a λ^+-Aronszajn tree T s.t. $\mathbb{P}(T)$ is λ-distributive.*

To sum up

A problem of a similar flavor

- Jensen constructed a λ^+-Souslin tree from $\mathsf{GCH} + \square_\xi(\lambda^+)$ with $\xi = \lambda$, and we relaxed it to $\xi = \lambda^+$.
- Ben-David and Shelah constructed a non-collapsing λ^+-Aronszajn tree from $\mathsf{GCH} + \square_\xi(\lambda^+, <\lambda^+)$ with $\xi = \lambda$, and we want to relax it to $\xi = \lambda^+$.

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The constructions under $\xi = \lambda$ use this assumption crucially:

- Jensen exploits the fact that $\square_\lambda(\lambda^+)$ yields a non-reflecting stationary set S. The definition of limit level T_α for $\alpha \in S$ involves throwing away many canonical limits from $\bigcup_{\beta<\alpha} T_\beta$.

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  This does not jam the later stages of the construction, since (one can arrange that) $\mathrm{acc}(C_\alpha) \cap S = \emptyset$ for all α.

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  This does not jam the later stages of the construction, since they build a λ-splitting tree, while $|C_\alpha| < \lambda$ for all α.

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A problem of a similar flavor

- Jensen constructed a λ^+-Souslin tree from $\mathsf{GCH} + \Box_\xi(\lambda^+)$ with $\xi = \lambda$, and we relaxed it to $\xi = \lambda^+$.
- Ben-David and Shelah constructed a non-collapsing λ^+-Aronszajn tree from $\mathsf{GCH} + \Box_\xi(\lambda^+, < \lambda^+)$ with $\xi = \lambda$, and we want to relax it to $\xi = \lambda^+$.

The constructions under $\xi = \lambda$ use this assumption crucially.

So, "relaxing $\xi = \lambda$ to $\xi = \lambda^+$", in fact, amounts to finding a different construction.

Same same, but different

Coherent Souslin trees

Exercise

Suppose that $\diamondsuit(\kappa)$ holds, and there exists a $\square_\kappa(\kappa)$-sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ satisfying the following:

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Then there exists a κ-Souslin tree.

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For a quick proof

See "How to construct a Souslin tree the right way" on my webpage.

Coherent Souslin trees

Proposition (Brodsky-Rinot, 2017)

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Note

Wlog, the A_i's are pairwise disjoint. Therefore, $|C_\alpha| = |\alpha|$.

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Then there exists a coherent κ-Souslin tree.

About the proof

Uses the microscopic approach for Souslin-tree constructions.

Distributive Aronszajn trees

Proposition (Brodsky-Rinot, 201∞)

Suppose that $\diamondsuit(\kappa)$ holds, and there exists a $\square_\kappa(\kappa, <\kappa)$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ satisfying the following:

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  a limit $\alpha < \kappa$ such that $\sup(\mathrm{nacc}(C) \cap E) = \alpha$ for all $C \in \mathcal{C}_\alpha$.

Recall

$\mathcal{C}_\alpha := \{ C_\beta \cap \alpha \mid \beta < \kappa, \sup(C_\beta \cap \alpha) = \alpha \}$.

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Note

Ben-David and Shelah used $\diamondsuit(\kappa)$ to seal cofinal branches.
We use club-guessing, instead.
(Instead of throwing away canonical limits, we inject noise)

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- For every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, there is a limit $\alpha < \kappa$ such that $\sup(\mathrm{nacc}(C_\alpha) \cap A_i) = \alpha$ for all $i < \theta$.

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About the proof

Uses walks on ordinals.

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Then $T(\vec{C})$ is θ-distributive.

About the proof

Uses walks on ordinals.
From $\vec{C}$, we cook up $\vec{D}$, and then the tree $T(\vec{C})$ is $\mathcal{T}(\rho_0^{\vec{D}})$.

To sum up

There are a few machines that take $\square_\xi(\kappa, <\mu)$-sequences $\vec{C}$ as inputs, and produce corresponding trees $T(\vec{C})$ as outputs.
We already mentioned two:

- The microscopic approach for Souslin-tree constructions;
- Walks on ordinals.

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There are a few machines that take $\square_\xi(\kappa, <\mu)$-sequences $\vec{C}$ as inputs, and produce corresponding trees $T(\vec{C})$ as outputs.
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To sum up

There are a few machines that take $\square_\xi(\kappa, < \mu)$-sequences $\vec{C}$ as inputs, and produce corresponding trees $T(\vec{C})$ as outputs.
We already mentioned two:

- The microscopic approach for Souslin-tree constructions;
- Walks on ordinals.

Whether the outcome tree $T(\vec{C})$ is Aronszajn/Souslin/Collapsing... depends on further features of $\vec{C}$.
So, if we were to use these machines, then we have to find a way to improve the $\vec{C}$'s.

Improve your square

Postprocessing functions

So, someone provides us with a raw $\square_\xi(\kappa, <\mu)$-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$. How do we proceed?

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By convention, let $\Phi(x) := \{\sup(x)\}$ for all $x \in \mathcal{P}(\kappa) \setminus \mathcal{K}(\kappa)$.

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Lemma (Brodsky-Rinot, 201∞)

If $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ *is a* $\Box_\xi(\kappa, <\mu)$*-sequence, and* $\min\{\xi, \mu\} < \kappa$*, then* $\vec{C}^\Phi := \langle \Phi(C_\alpha) \mid \alpha < \kappa\rangle$ *is a* $\Box_\xi(\kappa, <\mu)$*-sequence, as well.*

Postprocessing functions (cont.)

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This means that we can start with an arbitrary square sequence $\vec{C}$; then move to $\vec{C}^{\Phi_0}$, and then to $\vec{C}^{\Phi_1 \circ \Phi_0}$, and hopefully, after finitely many steps, we will end up with a useful sequence $\vec{C}^{\Phi_n \circ \cdots \circ \Phi_0}$.

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Question

What kind of postprocessing functions are there?

List of postprocessing functions

Postprocessing functions - example #1

Recall (postprocessing function)

A map $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ satisfying for all $x \in \mathcal{K}(\kappa)$:

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For all $x \in \mathcal{K}(\kappa)$, let:

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Well, the preceding doesn't quite work. Here is how it's done:

$$\Phi(x) := \begin{cases} \mathrm{acc}(x), & \text{if } \sup(\mathrm{acc}(x)) = \sup(x); \\ \end{cases}$$

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Postprocessing functions - example #2

For some fixed $\epsilon < \kappa$:

$$\Phi(x) := \begin{cases} \{\alpha \in x \mid \mathrm{otp}(x \cap \alpha) > \epsilon\}, & \text{if } \mathrm{otp}(x) > \epsilon; \\ x, & \text{otherwise.} \end{cases}$$

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More generally, for a fixed closed subset Σ of κ:

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Applications

A clever choice of Σ could transform a $\square_\xi(\kappa, < \mu)$-sequence into a $\square_{\xi'}(\kappa, < \mu')$-sequence with $\xi' < \xi$ or $\mu' < \mu$.

Postprocessing functions - example #3

For some fixed club $D \subseteq \kappa$:

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Another useful option:

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Applications

A clever choice of D could equip a $\square_\xi(\kappa, < \mu)$-sequence with some club-guessing features.

Postprocessing functions - example #4

For some fixed $A \subseteq \kappa$:

$$\Phi(x) := \begin{cases} \mathrm{cl}(\mathrm{nacc}(x) \cap A), & \text{if } \sup(\mathrm{nacc}(x) \cap A) = \sup(x); \\ x \setminus \sup(\mathrm{nacc}(x) \cap A), & \text{otherwise.} \end{cases}$$

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Applications

A dichotomy argument could provide A that would transform a $\Box_\xi(\kappa, < \mu)$-sequence into a $\Box_{\xi'}(\kappa, < \mu)$-sequence with $\xi' < \xi$.

Postprocessing functions - example #5

Theorem (Brodsky-Rinot, 201∞)

Suppose that $2^\lambda = \lambda^+$, $S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ *is stationary, and* $\langle C_\alpha \mid \alpha \in S\rangle$ *is a sequence such that each* C_α *is a club in* α *of order-type* $< \alpha$. *Then there exists a postprocessing function* $\Phi : \mathcal{K}(\lambda^+) \to \mathcal{K}(\lambda^+)$ *satisfying the following.*

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For every cofinal $A \subseteq \lambda^+$, *there exist stationarily many* $\alpha \in S$ *s.t.:*

1. $\mathrm{nacc}(\Phi(C_\alpha)) \subseteq A$;
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Postprocessing functions - example #5

Theorem (Brodsky-Rinot, 201∞)

Suppose that $2^\lambda = \lambda^+$, $S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is stationary, and $\langle C_\alpha \mid \alpha \in S \rangle$ is a sequence such that each C_α is a club in α of order-type $< \alpha$. Then there exists a postprocessing function $\Phi : \mathcal{K}(\lambda^+) \to \mathcal{K}(\lambda^+)$ satisfying the following.

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For λ singular, if $2^\lambda = \lambda^+$ and $\square^*_\lambda$ holds, then $\diamondsuit(S)$ holds for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects stationarily often.

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Not enough for intended applications

Hitting a single cofinal set A is nice, but we need to hit many A_i's.

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Lemma (Brodsky-Rinot, 201∞)

Assume $\diamondsuit(\kappa)$. Then there is a postprocessing $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ such that every sequence $\langle A_i \mid i < \kappa\rangle$ of cofinal subsets of κ may be encoded by a single stationary set G.

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Postprocessing functions - example #5

Corollary (Brodsky-Rinot, 201∞)

Suppose $\langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\square_\xi(\kappa, <\mu)$-sequence, and $2^{|\xi|} = \kappa$. For cofinally many $\theta < |\xi|$, there exists a postprocessing function $\Phi_\theta : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ satisfying the following. For every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, there are stat. many $\alpha < \kappa$ s.t. $\sup(\mathrm{nacc}(\Phi_\theta(C_\alpha)) \cap A_i) = \alpha$ for all $i < \theta$.

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Next problem

Each θ has its own Φ_θ. We need to integrate them together!

Postprocessing functions - example #5

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Remark

A statement parallel to the preceding, obtained by replacing $\xi < \kappa$ with $\mu < \kappa$ holds true as well.
(The proof, however, is entirely different)

Mixing postprocessing functions

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Mixing lemma (Brodsky-Rinot, 201∞)

Suppose $\langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\square_\xi(\kappa, <\mu)$-sequence, $\min\{\xi, \mu\} < \kappa$. For every $\Theta \subseteq \kappa$ and every sequence $\langle S_\theta \mid \theta \in \Theta \rangle$ of stationary subsets of κ, there is a postprocessing function $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ such that, for cofinally many $\theta \in \Theta$,

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is stationary.

This means

To each θ such that $\hat{S}_\theta$ is stationary, we may find a corresponding postprocessing function Φ_θ, and then we can mix them together letting $\Phi'(x) = \Phi_\theta(x)$ iff $\min(\Phi(x)) = \theta$.

An application

Conjecture

For every uncountable cardinal λ, if $\mathsf{GCH} + \square_{\lambda^+}(\lambda^+, < \lambda^+)$ holds, then there is a λ^+-Aronszajn tree T s.t. $\mathbb{P}(T)$ preserves cardinals.

Theorem (Brodsky-Rinot, 201∞)

For every singular cardinal λ, if $\mathsf{GCH} + \square_{\lambda^+}(\lambda^+, < \lambda)$ holds, then there is a λ^+-Aronszajn tree T s.t. $\mathbb{P}(T)$ is λ-distributive.

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Corollary

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An unrelated application of the mixing lemma

If $\square(\kappa)$ holds, then any fat subset of κ may be split into κ many fat sets.

Blowing up

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Theorem (Brodsky-Rinot, 201∞)

Assume GCH, λ is a singular cardinal, and there is a non-reflecting stationary subset of $E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$.
If $\square^*_\lambda$ holds, then there is a $\square_{\lambda^2}(\lambda^+, <\lambda^+)$-sequence $\vec{C}$, for which *the microscopic approach to Souslin-tree constructions* produces a λ^+-Souslin tree which is moreover free.