

ON FUNCTIONS THAT ARE ALMOST CONTINUOUS AND PERFECTLY EVERYWHERE SURJECTIVE BUT NOT JONES. LINEABILITY AND ADDITIVITY

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ABSTRACT. We show that the class of functions that are perfectly everywhere surjective and almost continuous in the sense of Stallings but are not Jones functions is \mathfrak{c}^+ -lineable. Moreover, it is consistent that this class is $2^{\mathfrak{c}}$ -lineable. We also prove that the additivity number for this class is between ω_1 and \mathfrak{c} . This lower bound can be achieved even when $\omega_1 < \mathfrak{c}$, as it is implied by the Covering Property Axiom CPA. The main step in this proof is the following theorem, which is of independent interest: CPA implies that there exists a family $\mathcal{F} \subset C(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that for every $g \in C(\mathbb{R})$ the set $g \setminus \bigcup \mathcal{F}$ has cardinality less than \mathfrak{c} . Some open problems are posed as well.

1. INTRODUCTION

During a Math conference in Kent State University (Kent, OH) in November of 2016 (and even more recently, in [17]) the following question was posed to the public:

How “large” (in terms of algebraic genericity) is the class of functions in $\mathbb{R}^{\mathbb{R}}$ that are perfectly everywhere surjective and almost continuous (in the sense of Stallings) but not Jones?

This question becomes clear once we define the following concepts.

Definition 1.1. Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$, we say that:

- (I) f is *perfectly everywhere surjective* ($f \in \text{PES}$) if $f[P] = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}$.

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- (II) f is a *Jones function* ($f \in \mathbf{J}$) if $C \cap f \neq \emptyset$ for every closed $C \subset \mathbb{R}^2$ with $\text{dom } C$ (i.e., projection of C on the first coordinate) has cardinality continuum \mathfrak{c} .
- (III) f is *almost continuous* (in the sense of Stallings; $f \in \mathbf{AC}$) if for each open set $G \subset \mathbb{R}^2$ such that $f \subset G$ there exists a continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $g \subset G$.

The notion of “being large” in terms of algebraic genericity is nowadays expressed in the following more precise terminology (see, e.g., [1, 3, 4, 11, 12, 19, 23, 26]).

Definition 1.2. Given a (finite or infinite) cardinal number κ , a subset M of a vector space X is called κ -*lineable* in X if there exists a linear space $Y \subset M \cup \{0\}$ of dimension κ .

Intuitively, lineability seeks for a linear structure within $M \cup \{0\}$ of the highest possible dimension. However, there exist sets M containing no linear substructures of highest dimension, [4]. Due to the previous reason, this “maximal lineability number” is best expressed as the lineability coefficient \mathcal{L} defined as the least cardinal for which there is *no* linear substructure of that cardinality (see [13] or [7].)

Definition 1.3. The *lineability coefficient* of a class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is defined as

$$\mathcal{L}(\mathcal{F}) = \min\{\kappa:$$

there is no κ -dimensional vector space V with $V \subset \mathcal{F} \cup \{0\}\}$.

Recall that \mathcal{F} admits the maximal lineability number if, and only if, $\mathcal{L}(\mathcal{F})$ is a cardinal successor, that is, $\mathcal{L}(\mathcal{F})$ is of the form κ^+ . (The symbol κ^+ stands for the successor cardinal of κ .) We refer the interested reader to [4, 8–10, 16, 21] for many applications of this concept to several different fields within mathematics and, for a complete modern state of the art of this area of research, see [1, 11].

On the other hand, and since the appearance of the work [20], the notion of lineability has been linked to that of the additivity coefficient A , which was introduced by the third author in [24, 25].

Definition 1.4. Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. The additivity of \mathcal{F} is defined as the following cardinal number:

$$A(\mathcal{F}) = \min(\{ |F| : F \subset \mathbb{R}^{\mathbb{R}} \wedge (\forall g \in \mathbb{R}^{\mathbb{R}})(g + F \not\subset \mathcal{F}) \} \cup \{(2^{\mathfrak{c}})^+\}).$$

The class of all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ is denoted by $\mathbf{C}(\mathbb{R})$. Recall also that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is almost continuous if and only if it intersects every *blocking set*, that is, a closed set $K \subset \mathbb{R}^2$ which meets

every continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$ and is disjoint with at least one function from $\mathbb{R}^{\mathbb{R}}$. The domain $\text{dom}(K)$ of every blocking set contains a non-degenerate connected set (see [22] or [24].)

It is known that the class J is a proper subclass both of the class PES and the class AC (although, until the present work, it has not been studied if it is a proper class of $\text{AC} \cap \text{PES}$). It is known that the family J , and so also each of the families PES and AC , is $2^{\mathfrak{c}}$ -lineable (see [19] and [18], respectively.)

This paper is arranged in two main sections. Section 2 focuses on answering the question mentioned earlier in this Introduction. Namely, we show that $\text{AC} \cap \text{PES} \setminus \text{J}$ is \mathfrak{c}^+ -lineable. On the other hand, it is known that $A(\text{PES}) = A(\text{J}) = A(\text{AC}) > \mathfrak{c}$ [20, Theorem 3.16]. In Section 3 we will provide lower and upper bounds for the additivity of $\text{AC} \cap \text{PES} \setminus \text{J}$. Some open questions and directions of research are also provided.

2. LINEABILITY OF THE CLASS $\text{AC} \cap \text{PES} \setminus \text{J}$

Let us recall the notion of *Bernstein set*. We say that $B \subset \mathbb{R}$ is a Bernstein set if B and $\mathbb{R} \setminus B$ meet each perfect set $P \subset \mathbb{R}$. Clearly, each Bernstein set can be decomposed into \mathfrak{c} -many Bernstein sets. Moreover, if B is a Bernstein set and P is a perfect set, then $|B \cap P| = \mathfrak{c}$, hence if B is Bernstein and $|C| < \mathfrak{c}$ then $B \setminus C$ and $B \cup C$ are Bernstein sets, too. Observe that $f \in \text{PES}$ if, and only if, each level set $f^{-1}(y)$, $y \in \mathbb{R}$, is a Bernstein set.

Lemma 2.1. *Let \mathcal{F} be the family of all closed subsets of \mathbb{R}^2 such that each $S \in \mathcal{F}$ is either a blocking set or equal to $P \times \{y\}$ for some perfect set $P \subset \mathbb{R}$ and $y \in \mathbb{R}$. For every Bernstein set $B \subset \mathbb{R}$ and a $C \subset \mathbb{R} \setminus \{0\}$ nowhere dense in \mathbb{R} there exists a function $\varphi \in \mathbb{R}^{\mathbb{R}}$ such that:*

- (I) $\varphi(x) = 0$ for every $x \in \mathbb{R} \setminus B$.
- (II) For every $\lambda \in \mathbb{R} \setminus \{0\}$, the set $\{x \in C: \varphi(x) = \lambda x\}$ has at most one element.
- (III) For every $S \in \mathcal{F}$, the set $\{x \in B: \langle x, \varphi(x) \rangle \in S\}$ has cardinality \mathfrak{c} .

In particular, $\varphi \in \text{AC} \cap \text{PES} \setminus \text{J}$.

Proof. Clearly the family \mathcal{F} has cardinality \mathfrak{c} . Let $\langle S_\xi: \xi < \mathfrak{c} \rangle$ be an enumeration of \mathcal{F} with each $S \in \mathcal{F}$ appearing \mathfrak{c} -many times.

By transfinite induction, on $\xi < \mathfrak{c}$, we will construct a sequence $\langle \langle x_\xi, y_\xi \rangle \in B \times \mathbb{R}: \xi < \mathfrak{c} \rangle$ aiming for $\varphi = \{\langle x_\xi, y_\xi \rangle: \xi < \mathfrak{c}\} \cup (X \times \{0\})$, where $X = \mathbb{R} \setminus \{x_\xi: \xi < \mathfrak{c}\}$.

So, assume that, for some $\xi < \mathfrak{c}$, the sequence $\langle \langle x_\zeta, y_\zeta \rangle : \zeta < \xi \rangle$ is already constructed. We need to choose $\langle x_\xi, y_\xi \rangle$. For this, consider two cases. If S_ξ is a blocking set, we choose

$$x_\xi \in \text{dom}(S_\xi) \cap B \setminus (C \cup \{x_\zeta : \zeta < \xi\}).$$

The choice is possible, since $\text{dom}(S_\xi) \setminus C$ has non-empty interior and so $\text{dom}(S_\xi) \cap B \setminus C$ has cardinality \mathfrak{c} .

Otherwise, $S_\xi = P \times \{y\}$ for some perfect set P . If $y = 0$ let $L_\xi = \emptyset$. If not, let L_ξ be the set of all $x \in P$ for which the line through $\langle 0, 0 \rangle$ and $\langle x, y \rangle$ intersects the set $\{\langle x_\zeta, y_\zeta \rangle : \zeta < \xi\}$. Then, L_ξ has cardinality smaller than \mathfrak{c} and we can choose

$$x_\xi \in \text{dom}(S_\xi) \cap B \setminus (L_\xi \cup \{x_\zeta : \zeta < \xi\}).$$

In either case we choose y_ξ so that $\langle x_\xi, y_\xi \rangle \in S_\xi$.

The above construction ensures that the sequence $\langle x_\xi : \xi < \mathfrak{c} \rangle$ is one-to-one, so our φ is indeed a function.

Clearly (i) holds, as $\{x_\xi : \xi < \mathfrak{c}\} \subseteq B$. The property (iii) holds, since each $S \in \mathcal{F}$ appears in our enumeration, as S_ξ , \mathfrak{c} -many times and $\langle x_\xi, \varphi(x_\xi) \rangle = \langle x_\xi, y_\xi \rangle \in S_\xi = S$. Finally, notice that our inductive step preserves (ii). Indeed, if S_ξ is a blocking set, then this is obvious, since then $x_\xi \notin C$. Otherwise, this is ensured by our choice of x_ξ outside of the set L_ξ .

To finish the proof we need to show that $\varphi \in \text{AC} \cap \text{PES} \setminus \text{J}$. Indeed, condition (iii) immediately implies that $\varphi \in \text{AC} \cap \text{PES}$. To see that $\varphi \notin \text{J}$ note that, by (ii), the closed set $Z_1 = \{\langle x, x \rangle : x \in C\}$ intersects φ in at most one point. Therefore, it contains an uncountable closed subset which does not intersect φ , justifying $\varphi \notin \text{J}$. \square

Proposition 2.2. *For every cardinal $\kappa < \mathfrak{c}$, the family $\text{AC} \cap \text{PES} \setminus \text{J}$ is 2^κ -lineable.*

Proof. We can assume, without loss of generality, that $\omega \leq \kappa < \mathfrak{c}$. Let $\{B_\alpha : \alpha < \kappa\}$ a partition of \mathbb{R} into κ many Bernstein sets. Let $C \subset \mathbb{R} \setminus \{0\}$ be an uncountable compact nowhere dense in \mathbb{R} . For example C can be a translation of the Cantor ternary set. For every $\alpha < \kappa$, let φ_α be the function provided by Lemma 2.1 with $B = B_\alpha$ and C as above. Consider the map $\Phi : \mathbb{R}^\kappa \rightarrow \mathbb{R}^\mathbb{R}$ defined by

$$\Phi(\langle a_\beta \rangle_{\beta < \kappa}) = \sum_{\beta < \kappa} a_\beta \varphi_\beta.$$

This sum is well defined since, for every $x \in \mathbb{R}$, $\varphi_\alpha(x) \neq 0$ only if $x \in B_\alpha$. Clearly Φ is a linear injection. Thus, the range of Φ is a linear space of dimension 2^κ . In fact, this is immediate if we prove

that $\dim(\mathbb{R}^\kappa) \geq \mathfrak{c}$, as for linear spaces over \mathbb{R} of dimension not smaller than \mathfrak{c} their dimension is equal to their cardinality. Now, as $\kappa \geq \omega$, \mathbb{R}^κ contains a subspace isomorphic with \mathbb{R}^ω . Since the dimension of \mathbb{R}^ω is equal to \mathfrak{c} (if \mathcal{A} is a family of infinite almost disjoint subsets of ω of cardinality \mathfrak{c} , then their characteristic functions are linearly independent), we obtain that the dimension of \mathbb{R}^κ is, at least, \mathfrak{c} .

Thus, to finish the proof, it is enough to show that $\Phi(\langle a_\beta \rangle_{\beta < \kappa}) \in \text{AC} \cap \text{PES} \setminus \text{J}$ whenever $a_\beta \neq 0$ for some $\beta < \kappa$.

Indeed, in such case, $\Phi(\langle a_\beta \rangle_{\beta < \kappa}) \upharpoonright B_\beta = a_\beta \varphi_\beta \upharpoonright B_\beta$ intersects every $S \in \mathcal{F}$, ensuring that $\Phi(\langle a_\beta \rangle_{\beta < \kappa}) \in \text{AC} \cap \text{PES}$.

To see that $\Phi(\langle a_\beta \rangle_{\beta < \kappa}) \notin \text{J}$ it is enough to show that its intersection with $Z_1 = \{\langle x, x \rangle : x \in C\}$ has cardinality less than \mathfrak{c} , as then there is a closed uncountable subset of Z_1 that does not intersect $\Phi(\langle a_\beta \rangle_{\beta < \kappa})$. But this is the case, since

$$Z_1 \cap \Phi(\langle a_\beta \rangle_{\beta < \kappa}) = \bigcup_{\beta < \kappa} Z_1 \cap a_\beta \varphi_\beta$$

is a union of $\kappa < \mathfrak{c}$ sets $Z_1 \cap a_\beta \varphi_\beta$ each having at most one element. (Indeed, $Z_1 \cap a_\beta \varphi_\beta = \emptyset$ for $a_\beta = 0$ and, for $a_\beta \neq 0$, its domain is contained in the set $\{x \in C : \varphi_\beta(x) = \frac{1}{a_\beta} x\}$ which, by (II), has at most one element.) \square

Since $2^\omega = \mathfrak{c}$, we obtain immediately from Proposition 2.2 that the family $\text{AC} \cap \text{PES} \setminus \text{J}$ is \mathfrak{c} -lineable. Note, however, that if we accept the set-theoretical assumption that \mathfrak{c} is singular, then we reach a little further. Indeed, in this case, we have $\text{cf } \mathfrak{c} < \mathfrak{c}$ and so $\text{AC} \cap \text{PES} \setminus \text{J}$ is $2^{\text{cf } \mathfrak{c}}$ -lineable. Thus, using the classical König's Theorem that $2^{\text{cf } \mathfrak{c}} > \mathfrak{c}$, we conclude that $\text{AC} \cap \text{PES} \setminus \text{J}$ is at least \mathfrak{c}^+ -lineable.

Next, we turn our attention to the case when \mathfrak{c} is regular. We start with the following proposition.

Proposition 2.3. *Assume that there exists a family $\{B_\alpha : \alpha < \kappa\}$ of almost disjoint Bernstein sets in \mathbb{R} , that is, such that the intersection of every two different sets from it has cardinality less than \mathfrak{c} . Then, the family $\text{AC} \cap \text{PES} \setminus \text{J}$ is κ -lineable.*

Proof. By Proposition 2.2, and the above remark, we can assume that $\kappa \geq \omega$. (Even that $\kappa > \mathfrak{c}$.) Let $C \subset \mathbb{R} \setminus \{0\}$ be an uncountable compact nowhere dense in \mathbb{R} . For every $\alpha < \kappa$, let φ_α be the function provided by Lemma 2.1 with $B = B_\alpha$ and C as above. It is enough to prove that the linear space generated by $\{\varphi_\alpha : \alpha < \kappa\}$ proves κ -lineability of $\text{AC} \cap \text{PES} \setminus \text{J}$. But any non-zero function from this space, of the form $f = \sum_{\alpha < \kappa} a_\alpha \varphi_\alpha = \sum_{i < n} a_{\alpha_i} \varphi_{\alpha_i}$, clearly belongs to $\text{AC} \cap \text{PES}$, the

argument being as in Proposition 2.2. Also, using the notation as in Proposition 2.2, $Z_1 \cap f$ has cardinality less than \mathfrak{c} , since it is contained in the union of two sets: the finite set $\bigcup_{i < n} Z_1 \cap a_{\alpha_i} \varphi_{\alpha_i}$ and the set $\bigcup_{i < j < n} B_{\alpha_i} \cap B_{\alpha_j}$ of cardinality less than \mathfrak{c} . So, as in Proposition 2.2, $f \notin J$. \square

We will also need the following simple fact.

Lemma 2.4. *If \mathfrak{c} is regular, then there exists a family of cardinality \mathfrak{c}^+ of almost disjoint Bernstein sets in \mathbb{R} .*

Proof. First observe that for any family $\{B_\alpha : \alpha < \mathfrak{c}\}$ of almost disjoint Bernstein sets there exists a Bernstein set B which is almost disjoint with every set B_α , $\alpha < \mathfrak{c}$. In fact, let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be a family of all perfect sets on \mathbb{R} . For every $\alpha < \mathfrak{c}$ choose, using a fact that \mathfrak{c} is regular, a point

$$x_\alpha \in P_\alpha \cap \left(B_\alpha \setminus \bigcup_{\beta < \alpha} B_\beta \right).$$

Let $B = \{x_\alpha : \alpha < \mathfrak{c}\}$. Then B meets each perfect set. Moreover for every $\alpha < \mathfrak{c}$,

$$B \cap B_\alpha \subset \{x_\beta : \beta \leq \alpha\}$$

is of size less than \mathfrak{c} , hence B is almost disjoint with B_α . Finally, since B_0 is a Bernstein set, $|B \cap B_0| < \mathfrak{c}$, and $|B_0 \cap P| = \mathfrak{c}$ for any perfect set P , $\mathbb{R} \setminus B$ meets P . Therefore, B is a Bernstein set.

Now, Kuratowski-Zorn's Lemma yields that there exists a maximal, with respect to the inclusion, family \mathcal{B} of almost disjoint Bernstein sets. By the remark above, $|\mathcal{B}| \geq \mathfrak{c}^+$. \square

Theorem 2.5. *$\text{AC} \cap \text{PES} \setminus J$ is \mathfrak{c}^+ -lineable.*

Proof. The case when \mathfrak{c} is singular follows from Proposition 2.2 and the subsequent short discussion. The case when \mathfrak{c} is regular follows from Proposition 2.3 and Lemma 2.4. \square

Theorem 2.6. *If $2^{<\mathfrak{c}} = \mathfrak{c}$, then $\text{AC} \cap \text{PES} \setminus J$ is $2^\mathfrak{c}$ -lineable. Hence*

$$\mathcal{L}(\text{AC} \cap \text{PES} \setminus J) = (2^\mathfrak{c})^+.$$

Proof. By Proposition 2.3, it is enough to show that $2^{<\mathfrak{c}} = \mathfrak{c}$ implies that there exists a family of cardinality $2^\mathfrak{c}$ of almost disjoint Bernstein sets on \mathbb{R} . Let T be a tree of all 0-1 sequences $t: \alpha \rightarrow 2$, $\alpha < \mathfrak{c}$. Let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be a sequence of all perfect subsets of \mathbb{R} . For every $t \in T$ let $\text{dom}(t)$ denote the domain of t . Note that $|T| = \mathfrak{c}$. Thus, we can define inductively a function $F: T \rightarrow \mathbb{R}$ such that

$$(I) \quad F(t) \in P_{\text{dom}(t)};$$

(II) F is one-to-one.

For every $f: \mathfrak{c} \rightarrow 2$ let

$$S_f = F[\{t \in T : t \subset f\}].$$

Clearly, $|S_f \cap S_g| < \mathfrak{c}$ if $f \neq g$. For a given perfect set P_α and $f: \mathfrak{c} \rightarrow 2$ we have $F(f \upharpoonright \alpha) \in S_f \cap P_\alpha$. Hence S_f is a Bernstein set. In particular, the family $\{S_f : f \in 2^\mathfrak{c}\}$ satisfies the assertion. \square

Corollary 2.7. *If $2^{<\mathfrak{c}} = \mathfrak{c}$, then the classes $\text{PES} \setminus \text{J}$ and $\text{AC} \setminus \text{J}$ are $2^\mathfrak{c}$ -lineable. In particular,*

$$\mathcal{L}(\text{PES} \setminus \text{J}) = (2^\mathfrak{c})^+ = \mathcal{L}(\text{AC} \setminus \text{J}).$$

Clearly, the continuum hypothesis, CH, (as well as the Martin's axiom MA) implies that $2^{<\mathfrak{c}} = \mathfrak{c}$. Hence, $2^\mathfrak{c}$ -lineability of the class $\text{AC} \cap \text{PES} \setminus \text{J}$ is consistent with ZFC. By Theorem 2.5, $2^\mathfrak{c}$ -lineability of $\text{AC} \cap \text{PES} \setminus \text{J}$ follows also from $2^\mathfrak{c} = \mathfrak{c}^+$.

Problem 2.8. *Are the families $\text{AC} \cap \text{PES} \setminus \text{J}$, $\text{PES} \setminus \text{J}$, and $\text{AC} \setminus \text{J}$ $2^\mathfrak{c}$ -lineable in ZFC? What about their algebraicity (see [2, 5, 6]), when considered within the class of complex functions?*

3. ADDITIVITY OF THE CLASSES $\text{AC} \cap \text{PES} \setminus \text{J}$ AND $\text{PES} \setminus \text{J}$

In this section we show that $\omega_1 \leq A(\text{AC} \cap \text{PES} \setminus \text{J}) \leq A(\text{PES} \setminus \text{J}) \leq \mathfrak{c}$ and that it is consistent with ZFC that the last inequality is strict. Of course, the inequality $A(\text{AC} \cap \text{PES} \setminus \text{J}) \leq A(\text{PES} \setminus \text{J})$ follows immediately from monotonicity of A operator. The upper bound is justified below. We start with showing the upper bound.

Theorem 3.1. $A(\text{PES} \setminus \text{J}) \leq \mathfrak{c}$.

Proof. Let $F = C(\mathbb{R})$. Since $|C(\mathbb{R})| = \mathfrak{c}$, it is enough to show that $h + C(\mathbb{R}) \not\subset \text{PES} \setminus \text{J}$ for every $h \in \mathbb{R}^\mathbb{R}$.

Indeed, by way of contradiction assume that $h + C(\mathbb{R}) \subset \text{PES} \setminus \text{J}$ for some $h \in \mathbb{R}^\mathbb{R}$. In particular, $h \in h + C(\mathbb{R}) \subset \text{PES} \setminus \text{J}$ and so $h \notin \text{J}$. Therefore, there exists a closed (even compact) $C \subset \mathbb{R}^2$ such that $|\text{dom } C| = \mathfrak{c}$ and $C \cap h = \emptyset$. The function $\gamma: \text{dom } C \rightarrow \mathbb{R}$ given by $\gamma(x) = \inf\{y: \langle x, y \rangle \in C\}$ is Borel (in fact, it is lower semi-continuous). So, there exists a perfect compact $P \subset \text{dom } C$ such that $\gamma \upharpoonright P$ is continuous. By Tietze's Extension Theorem, there exists an extension $f \in C(\mathbb{R})$ of $\gamma \upharpoonright P$. But then $0 \notin (h - f)[P]$, since h is disjoint with $C \supset \gamma \upharpoonright P$. Hence, $h - f$ does not belong to PES in contradiction with $h + C(\mathbb{R}) \subset \text{PES}$, what completes the proof. \square

Next, we will prove the lower bound: $A(\text{AC} \cap \text{PES} \setminus \text{J}) \geq \omega_1$. For this we will need the following lemma.

Lemma 3.2. *Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be countable. Then, there exists a perfect $Q \subset [0, 1]$ and a continuous function $\gamma: Q \rightarrow \mathbb{R}$ such that for every $f \in \mathcal{F}$ and $y \in \mathbb{R}$ the set $(\gamma + f \upharpoonright Q)^{-1}(y)$ does not contain a perfect subset.*

Proof. Let $\{f_n: n < \omega\}$ be an enumeration of \mathcal{F} . Let $\{I_s: s \in 2^{<\omega}\}$ be the family of closed subintervals of $[0, 1]$, each of length $3^{-|s|}$, used in the classical construction of the Cantor ternary set; that is, for every $s \in 2^{<\omega}$, I_{s0} and I_{s1} are, respectively, the left and the right components of I_s from which we removed its middle third. (Here $|t|$ denotes the length of t , that is, $|t| = n$ whenever $t \in 2^n$.)

We will construct, by induction on $n < \omega$, the family $\{P_s: s \in 2^{<\omega}\}$ of compact perfect subsets of $[0, 1]$ such that, for every $s \in 2^{<\omega}$,

$$P_{s0} \text{ and } P_{s1} \text{ are disjoint perfect subsets of } P_s.$$

We aim for $\gamma = \bigcap_{n < \omega} \bigcup_{s \in 2^n} P_s \times I_s$. Clearly γ defined like this is a continuous bijection from $Q = \bigcap_{n < \omega} \bigcup_{s \in 2^n} P_s$ into the Cantor ternary set.

We start with choosing a perfect set $P_\emptyset \subset [0, 1]$ such that either $f_0 \upharpoonright P_\emptyset$ is continuous, or there is no perfect subset P of P_\emptyset for which $f_0 \upharpoonright P$ is continuous. Also, if for some $s \in 2^n$ the perfect set P_s is already constructed, we choose the sets P_{s0} and P_{s1} as follows. First choose a perfect subset Q_s of P_s such that either $f_n \upharpoonright Q_s$ is continuous, or there is no perfect subset P of Q_s for which $f_n \upharpoonright P$ is continuous. Let $Z = \{i \leq n: f_i \upharpoonright Q_s \text{ is continuous}\}$. For every $i \in Z$ choose $\delta_i > 0$ such that for every $x, y \in Q_s$ with $|x - y| < \delta_i$, we have $|f_i(x) - f_i(y)| < 3^{-(n+1)}$. Let $\delta_s = \min(\{\delta_i: i \in Z\} \cup \{1\})$ and let P_{s0} and P_{s1} be disjoint perfect subsets of Q_s such that $P_{s0} \cup P_{s1}$ has the diameter smaller than δ_s . This finishes the inductive construction.

To see that γ constructed with such a sequence is as needed, fix an $n < \omega$ and $y \in \mathbb{R}$. It is enough to show that, for every $s \in 2^n$, the set

$$(\gamma + f_n \upharpoonright Q)^{-1}(y) \cap Q_s = \{x \in Q \cap Q_s: \gamma(x) = y - f_n(x)\}$$

does not contain a perfect subset. Indeed, this is clearly true when there is no perfect subset P of Q_s for which $f_n \upharpoonright P$ is continuous. So, assume that $f_n \upharpoonright Q_s$ is continuous. The proof is completed by noticing that, in such case, the set $\{x \in Q \cap Q_s: \gamma(x) = y - f_n(x)\}$ has at most one element. To see this, by way of contradiction assume that there are distinct $x_0, x_1 \in Q \cap Q_s$ for which $\gamma(x_0) = y - f_n(x_0)$ and $\gamma(x_1) = y - f_n(x_1)$. Then, $|\gamma(x_0) - \gamma(x_1)| = |f_n(x_0) - f_n(x_1)|$. However, this is impossible. Indeed, there exists $t \in 2^{<\omega}$ containing s such that $|t| \geq n$ and each of the sets P_{t0} and P_{t1} contains precisely

one of the points x_0, x_1 . But then, $|\gamma(x_0) - \gamma(x_1)| \geq 3^{-(|t|+1)}$ while $|f_n(x_0) - f_n(x_1)| < 3^{-(|t|+1)}$, since the number $|x_0 - x_1|$ is smaller than the diameter of $P_{t0} \cup P_{t1}$, so also smaller than δ_t . \square

Theorem 3.3. $\omega_1 \leq A(\text{AC} \cap \text{PES} \setminus \text{J})$.

Proof. Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be countable. We need to find an $h \in \mathbb{R}^{\mathbb{R}}$ such that $h + \mathcal{F} \subset \text{AC} \cap \text{PES} \setminus \text{J}$. We can assume that \mathcal{F} is an additive group. Let $\gamma: Q \rightarrow \mathbb{R}$ be as in Lemma 3.2. We may assume that Q is nowhere dense. Let $\{Q_f: f \in \mathcal{F}\}$ be pairwise disjoint perfect subsets of Q . We will find $h \in \mathbb{R}^{\mathbb{R}}$ such that, for every $f \in \mathcal{F}$, $h + f$ is disjoint with $\gamma \upharpoonright Q_f$. This will ensure that $h + f \notin \text{J}$.

Let $\{B, B'\}$ be a partition of $\mathbb{R} \setminus Q$ such that each of its elements meets every perfect set $P \subset \mathbb{R} \setminus Q$. Let $\langle \langle P_\xi, z_\xi, f_\xi \rangle : \xi < \mathfrak{c} \rangle$ be an enumeration of $\mathcal{P} \times \mathbb{R} \times \mathcal{F}$, where \mathcal{P} is the family of all perfect subsets of \mathbb{R} , and let $\langle \langle K_\xi, f'_\xi \rangle : \xi < \mathfrak{c} \rangle$ be an enumeration of $\mathcal{K} \times \mathcal{F}$, where \mathcal{K} is the family of all blocking sets in \mathbb{R}^2 . We will construct, by transfinite induction on $\xi < \mathfrak{c}$, two sequences $\langle \langle x_\xi, y_\xi \rangle : \xi < \mathfrak{c} \rangle$, $\langle \langle x'_\xi, y'_\xi \rangle : \xi < \mathfrak{c} \rangle$, of points in \mathbb{R}^2 such that for every $\xi < \mathfrak{c}$

- (I) $x_\xi \in P_\xi \setminus \{x_\zeta : \zeta < \xi\}$, and $x_\xi \in B$ if $P_\xi \cap \bigcup_{f \in \mathcal{F}} Q_f$ is countable,
- (II) $y_\xi + f_\xi(x_\xi) = z_\xi$,
- (III) $y_\xi + f(x_\xi) \neq \gamma(x_\xi)$ for every $f \in \mathcal{F}$ with $x_\xi \in Q_f$, and
- (IV) $x'_\xi \in \text{dom}(K_\xi) \cap B' \setminus \{x'_\zeta : \zeta < \xi\}$, and $\langle x'_\xi, y'_\xi + f'_\xi(x'_\xi) \rangle \in K_\xi$.

It is easy to see that, by (I), $h_0 = \{\langle x_\xi, y_\xi \rangle, \langle x'_\xi, y'_\xi \rangle : \xi < \mathfrak{c}\}$ is a partial function. By (III), we can extend h_0 to an $h \in \mathbb{R}^{\mathbb{R}}$ such that for every $f \in \mathcal{F}$, $h + f$ is disjoint with $\gamma \upharpoonright Q_f$, so that $h + f \notin \text{J}$. The condition $h + f \in \text{PES}$ is ensured by (I) and (II). Finally, (IV) implies $h + f \in \text{AC}$.

It remains to construct our sequence. For this assume that, for some $\xi < \mathfrak{c}$, the sequences $\langle x_\zeta, y_\zeta \rangle, \langle x'_\zeta, y'_\zeta \rangle, \zeta < \xi$, is already constructed.

First we choose $\langle x_\xi, y_\xi \rangle$. If there is no $f \in \mathcal{F}$ for which $P_\xi \cap Q_f$ is uncountable, then it is enough to pick $x_\xi \in P_\xi \cap B \setminus \{x_\zeta : \zeta < \xi\}$ not in the countable set $P_\xi \cap \bigcup_{f \in \mathcal{F}} Q_f$ and define $y_\xi = z_\xi - f_\xi(x_\xi)$. This ensures that conditions (I)-(III) are satisfied, (III) in void.

So, assume that there is an $f \in \mathcal{F}$ for which $P_\xi \cap Q_f$ is uncountable and let P be a perfect subset of $P_\xi \cap Q_f$ for such $f \in \mathcal{F}$. To ensure (I), we will choose $x_\xi \in P \setminus \{x_\zeta : \zeta < \xi\}$. Moreover, to ensure (II) and (III), we need to choose x_ξ so that $f_\xi(x_\xi) - f(x_\xi) \neq z_\xi - \gamma(x_\xi)$. If $\hat{f} = f_\xi - f \in \mathcal{F}$, then this last requirement can be written as $x_\xi \notin (\gamma + \hat{f} \upharpoonright Q)^{-1}(z_\xi)$. Now, by Lemma 3.2, the set $(\gamma + \hat{f} \upharpoonright Q)^{-1}(z_\xi)$ does not contain any perfect set. Therefore, $P \setminus (\gamma + \hat{f} \upharpoonright Q)^{-1}(z_\xi)$ must have cardinality \mathfrak{c} . In particular, we can choose $x_\xi \in P \setminus \left((\gamma + \hat{f} \upharpoonright Q)^{-1}(z_\xi) \cup \{x_\zeta : \zeta < \xi\} \right)$.

This choice, together with defining $y_\xi = z_\xi - f_\xi(x_\xi)$ ensures that the conditions (I)–(III) are satisfied.

Next, we choose $\langle x'_\xi, y'_\xi \rangle$. Let $K = K_\xi$ and $f = f'_\xi$. Since $\text{dom}(K)$ has non-empty interior and Q is nowhere dense, $B' \cap \text{dom}(K)$ is of size continuum, so we can choose $x'_\xi \in (B' \cap \text{dom}(K)) \setminus \{x'_\zeta : \zeta < \xi\}$. Fix $y \in \mathbb{R}$ such that $\langle x'_\xi, y \rangle \in K$ and put $y'_\xi = y - f(x'_\xi)$. This ensures that condition (IV) is satisfied. \square

Corollary 3.4. $\omega_1 \leq A(\text{AC} \cap \text{PES} \setminus \text{J}) \leq A(\text{PES} \setminus \text{J}) \leq \mathfrak{c}$. In particular, CH implies that $A(\text{AC} \cap \text{PES} \setminus \text{J}) = A(\text{PES} \setminus \text{J}) = \mathfrak{c}$.

The natural question here is, whether either the first or the last inequality in Corollary 3.4 can be replaced, in ZFC, by the equality. In what follows, we show that the last inequality can be strict. For this, we will need the following lemma, which is a modification of Theorem 3.1.

Lemma 3.5. *Let $\mathcal{F} \subset C(\mathbb{R})$ be such that for every $g \in C(\mathbb{R})$ the set $g \setminus \bigcup \mathcal{F}$ has cardinality less than \mathfrak{c} . If $|\mathcal{F}| < \mathfrak{c}$ and \mathcal{F} contains the constant zero function, then $h + \mathcal{F} \not\subset \text{PES} \setminus \text{J}$ for every $h \in \mathbb{R}^\mathbb{R}$. In particular, $A(\text{PES} \setminus \text{J}) \leq |\mathcal{F}|$.*

Proof. We proceed as in the proof of Theorem 3.1. That is, by way of contradiction assume that $h + \mathcal{F} \subset \text{PES} \setminus \text{J}$ for some $h \in \mathbb{R}^\mathbb{R}$. In particular, $h \in h + \mathcal{F} \subset \text{PES} \setminus \text{J}$ and so $h \notin \text{J}$. Therefore, there exists a compact $C \subset \mathbb{R}^2$ such that $|\text{dom } C| = \mathfrak{c}$ and $C \cap h = \emptyset$. The function $\gamma: \text{dom } C \rightarrow \mathbb{R}$ given by $\gamma(x) = \inf\{y: \langle x, y \rangle \in C\}$ is Borel. So, there exists a perfect compact $P \subset \text{dom } C$ such that $\gamma \upharpoonright P$ is continuous. By Tietze's Extension Theorem, there exists an extension $g \in C(\mathbb{R})$ of $\gamma \upharpoonright P$. So, $\gamma \setminus \bigcup \mathcal{F} \subset g \setminus \bigcup \mathcal{F}$ has cardinality less than \mathfrak{c} . In particular, there exists an $f \in \mathcal{F}$ such that the set $Q = \{x \in P: f(x) = \gamma(x)\}$ is closed and uncountable, hence it contains a perfect set. But then $0 \notin (h - f)[Q]$, since h is disjoint with $C \supset \gamma \upharpoonright Q$. Hence, $h - f$ does not belong to PES in contradiction with $h + \mathcal{F} \subset \text{PES}$, what completes the proof. \square

We will also need the following result, which is of the independent interest.

Theorem 3.6. *The Covering Property Axiom CPA implies that there exists a family $\mathcal{F} \subset C(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that for every $g \in C(\mathbb{R})$ the set $g \setminus \bigcup \mathcal{F}$ has cardinality less than \mathfrak{c} .*

Proof. Under CPA we have $\mathfrak{c} = \omega_2$. So, we just need to find a desired family \mathcal{F} of cardinality ω_1 . Notice, that it is enough to prove that for every compact perfect set $Z \subset \mathbb{R}$ there exists a family $\mathcal{F}_Z \subset C(\mathbb{R})$

of cardinality $\omega_1 < \mathfrak{c}$ such that for every $g \in C(Z)$ the set $g \setminus \bigcup \mathcal{F}_Z$ has cardinality less than \mathfrak{c} . Indeed, if this holds, then the family $\mathcal{F} = \bigcup_{n < \omega} \mathcal{F}_{[-n, n]}$ is as needed. So, fix a compact perfect set $Z \subset \mathbb{R}$.

It has been proved in [15] (see also [14, Thm. 4.1.1]) that there exists a covering \mathcal{K} of \mathbb{R}^2 of cardinality ω_1 by compact sets such that every $K \in \mathcal{K}$ has a one-to-one projection onto one of the coordinates. We will repeat here the same argument for the product $Z \times C(Z)$ in place of \mathbb{R}^2 . We consider $C(Z)$ with the uniform convergence topology, so that $Z \times C(Z)$ is a Polish space.

We will use terminology and notation as in [14]. Let \mathcal{E} be the family of all compact perfect subsets of $X = Z \times C(Z)$ such that every $P \in \mathcal{E}$ has a one-to-one projection onto either Z or $C(Z)$. Then, the argument precisely as the one used in the proof of [14, Prop. 4.1.3(b)] shows that the family \mathcal{E} is $\mathcal{F}_{\text{prism}}$ -dense. Therefore, by $\text{CPA}_{\text{prism}}$, there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \leq \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \leq \omega_1$.

Let \mathcal{F}_0 be the family of all $P \in \mathcal{E}_0$ for which the projection onto Z is one-to-one. Thus, every $P \in \mathcal{F}_0$ is a continuous function from a compact set $\text{dom}(P) \subset Z$ into $C(Z)$. Therefore, the map $f_P: \text{dom}(P) \rightarrow \mathbb{R}$ defined as $f_P(x) = P(x)(x)$ is continuous and, by Tietze's Extension Theorem, can be extended to $\hat{f}_P \in C(\mathbb{R})$. We claim that the family $\mathcal{F}_Z = \{\hat{f}_P: P \in \mathcal{F}_0\}$ is as needed.

To see this, fix a $g \in C(Z)$ and notice that

$$\begin{aligned} \text{dom}\left(g \cap \bigcup \mathcal{F}_Z\right) &= \bigcup_{P \in \mathcal{F}_0} \{x \in Z: g(x) = \hat{f}_P(x)\} \\ &\supseteq \bigcup_{P \in \mathcal{F}_0} \{x \in \text{dom}(P): g(x) = f_P(x)\} \\ &= \bigcup_{P \in \mathcal{F}_0} \{x \in \text{dom}(P): g(x) = P(x)(x)\} \\ &\supseteq \bigcup_{P \in \mathcal{F}_0} \{x \in \text{dom}(P): P(x) = g\} \\ &= \text{dom}\left((Z \times \{g\}) \cap \bigcup \mathcal{F}_0\right). \end{aligned}$$

We need to show that $Z \setminus \text{dom}(g \cap \bigcup \mathcal{F}_Z)$ has cardinality at most ω_1 . For this, it is enough to prove that $Z \setminus \text{dom}((Z \times \{g\}) \cap \bigcup \mathcal{F}_0)$ has cardinality at most ω_1 . But this is the case, since $(Z \times \{g\}) \setminus \bigcup \mathcal{F}_0$ has cardinality at most ω_1 , as it is contained in the union of the following two sets, each of cardinality $\leq \omega_1$: $X \setminus \bigcup \mathcal{E}_0$ and $\bigcup_{P \in \mathcal{E}_0 \setminus \mathcal{F}_0} ((Z \times \{g\}) \cap P)$, where each set $(Z \times \{g\}) \cap P$ contains at most one element, as the

projection of any $P \in \mathcal{E}_0 \setminus \mathcal{F}_0$ onto $C(Z)$ is one-to-one and $|\mathcal{E}_0 \setminus \mathcal{F}_0| \leq \omega_1$. \square

Clearly, the family $\mathcal{F} \subset C(\mathbb{R})$ from Theorem 3.6 is of cardinality less than \mathfrak{c} and has a property that any continuous g from a perfect set $Q \subset \mathbb{R}$ into \mathbb{R} agrees with some $f \in \mathcal{F}$ on a set of cardinality \mathfrak{c} . This property cannot be proved in ZFC. In fact, MA implies that for every perfect set $P \subset \mathbb{R}$ there is a perfect set $Q \subset P$ and a $g \in C(Q)$ such that $g \cap f = \emptyset$ for every $f \in \mathcal{F}$.¹

Corollary 3.7. *CPA implies that $A(\text{PES} \setminus \text{J}) \leq \omega_1$. In particular, it is consistent with ZFC that $A(\text{AC} \cap \text{PES} \setminus \text{J}) = A(\text{PES} \setminus \text{J}) = \omega_1 < \mathfrak{c}$.*

Proof. To see $A(\text{PES} \setminus \text{J}) \leq \omega_1$, let \mathcal{F} be as in Theorem 3.6. We can assume that \mathcal{F} contains the constant zero function, adding it to \mathcal{F} , if necessary. Then, by Lemma 3.5, $A(\text{PES} \setminus \text{J}) \leq |\mathcal{F}| = \omega_1$.

The additional statement follows from this, Corollary 3.4, and the fact that CPA implies $\mathfrak{c} = \omega_2$. \square

Problem 3.8. *Is either of the inequalities $\omega_1 < A(\text{AC} \cap \text{PES} \setminus \text{J})$ or $A(\text{AC} \cap \text{PES} \setminus \text{J}) < A(\text{PES} \setminus \text{J})$ consistent with ZFC? What about the consistency of $\omega_1 < A(\text{AC} \cap \text{PES} \setminus \text{J}) < A(\text{PES} \setminus \text{J}) < \mathfrak{c}$?*

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¹This follows from the fact that, under MA, the ideal of meager sets is \mathfrak{c} -additive. Indeed, this implies that $(P \times \mathbb{R}) \cap \bigcup \mathcal{F}$ is meager in $P \times \mathbb{R}$. So, by the Kuratowski-Ulam theorem, $(P \times \mathbb{R}) \setminus \bigcup \mathcal{F}$ contains a comeager horizontal section and so also a set g of the form $Q \times \{y\}$.

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