A THEORY OF PAIRS FOR NON-VALUATIONAL STRUCTURES

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ABSTRACT. Given a weakly o-minimal structure $\mathcal{M}$ and its o-minimal completion $\bar{\mathcal{M}}$, we first associate to $\bar{\mathcal{M}}$ a canonical language and then prove that $Th(\mathcal{M})$ determines $Th(\bar{\mathcal{M}})$. We then investigate the theory of the pair $(\bar{\mathcal{M}}, \mathcal{M})$ in the spirit of the theory of dense pairs of o-minimal structures, and prove, among other results, that it is near model complete, and every definable open subset of $\bar{M}^n$ is already definable in $\bar{\mathcal{M}}$.

We give an example of a weakly o-minimal structure which interprets $\bar{\mathcal{M}}$ and show that it is not elementarily equivalent to any reduct of an o-minimal trace.

1. INTRODUCTION

An expansion $\mathcal{M}$ of an ordered group is *weakly o-minimal non-valuational* (below we use "non-valuational" for short) if it is weakly o-minimal (every definable subset of M is a finite union of convex sets) and does not admit any definable non-trivial convex sub-groups. Non-valuational structures were introduced in [6] and more systematically studied in [10] and [11]. In those works Wencel showed that to a non-valuational structure $\mathcal{M}$ one can associate an o-minimal structure $\bar{\mathcal{M}}$, whose universe is $\bar{M}$ – the definable Dedekind completion of $\mathcal{M}$ – and with the additional property that the structure which $\bar{\mathcal{M}}$ induces on (the natural embedding of) M (in $\bar{\mathcal{M}}$) is precisely the structure $\mathcal{M}$. Wencel called the structure $\bar{\mathcal{M}}$ *the canonical o-minimal completion of* $\mathcal{M}$. In [5] Keren shows that $\bar{\mathcal{M}}$ has the same definable sets as the structure $\mathcal{M}^*$, whose atomic sets are all sets of the form $\mathrm{cl}_{\bar{M}}(S) \subseteq \bar{M}^n$ for $\mathcal{M}$-definable $S \subseteq M^n$, (see Proposition 2.7 below). Both Wencel and Keren's constructions have the problem that the signatures of the resulting structures depend on the structure $\mathcal{M}$, rather than on its signature.

In the present paper we address this problem by considering, for $A \subseteq M$, structures of the form $\mathcal{M}^*_A$ whose atomic sets are all sets of the form $\mathrm{cl}_{\bar{M}}(S)$ for S an $\mathcal{M}$-definable set over A. The starting point of the present work, and the main result of Section 2 is:

Theorem 1. *Let $\mathcal{M}$ be a non-valuational structure. Then $\mathcal{M}^*_\emptyset$ and $\mathcal{M}^*$ have the same definable sets. Moreover, if $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{M}^*_\emptyset \equiv \mathcal{N}^*_\emptyset$.*

This result shows that to a non-valuational theory T we can associate an o-minimal theory T^* which can be viewed as an invariant of T. Consequently, any of the o-minimal properties of T^* can reflect on the weakly o-minimal T and vice versa. This plays a crucial role in the proof of Theorem 3 below.

Section 5 is dedicated to the study of the theory of the pair $\mathcal{M}^P = (\mathcal{M}^*_\emptyset, \mathcal{M})$ for $\mathcal{M}$ non-valuational, in the spirit of van den Dries' study of o-minimal dense pairs (see [9]). Our main result is the following:

Theorem 2. *Let $\mathcal{M}$ be non-valuational.*

(1) *If $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{M}^P \equiv \mathcal{N}^P$.*
We let $T^P = Th(\mathcal{M}^P)$ and assume $\tilde{\mathcal{N}} = (\mathcal{N}', \mathcal{N}) \models T^P$.

(2) *If $Y \subseteq (N')^n$ is $\emptyset$-definable in $\tilde{\mathcal{N}}$ then it can be written as a boolean combination of sets defined by formulas of the form*

$$\exists x_1 \cdots \exists x_k \Big(\bigwedge_{i=1}^{k} x_i \in P \,\&\, \varphi(x_1, \ldots, x_k, y)\Big), \tag{1}$$

and $\varphi(x, y)$ is a formula of the o-minimal structure $\mathcal{M}'$.

(3) *If $X \subseteq N^k$ is definable in $\tilde{\mathcal{N}}$ over $A \subseteq N$ then X is already definable in the weakly o-minimal $\mathcal{N}$.*

(4) *If $U \subseteq (N')^k$ is a definable open set in $\tilde{\mathcal{N}}$ then U is already definable in the o-minimal structure $\mathcal{N}'$. In particular, $\tilde{\mathcal{N}}$ has an o-minimal open core.*

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The above results show that pairs $(\mathcal{M}', \mathcal{M})$ as above fit into the setting of recent works by Eleftherious, Gunaydin and Hieronymi (see for example [3]) on expansions of o-minimal structures by dense predicates.

Non-valuational structures arise naturally in the study of dense pairs of o-minimal structures. Namely, if $\mathcal{M} \prec \mathcal{N}$ are o-minimal expansions of ordered groups and M is dense in N then the structure induced on M from $\mathcal{N}$ is non-valuational (weak o-minimality follows from [1] and non-valuationality is easy, see e.g., [4]). Since every ordered group which is a reduct of a non-valuational structure, or even elementarily equivalent to one, is also such, a question arises whether every non-valuational structure arises in this manner.

First, some terminology. A non-valuational structure $\mathcal{M}$ is called an *o-minimal trace* if there is a dense pair $\mathcal{M}_0 \prec \mathcal{N}$ such that $\langle M_0, <\rangle = \langle M, <\rangle$ (i.e., the structures $\mathcal{M}_0$ and $\mathcal{M}$ have the same underlying ordered set) and the induced structure on M in the pair $(\mathcal{N}, \mathcal{M}_0)$ has the same definable sets as $\mathcal{M}$ (see [4] for details). In [4] we showed that an ordered reduct of a non-valuational o-minimal trace need not be an o-minimal trace itself, and that the class of reducts of o-minimal traces is not closed under elementary equivalence. In the present paper we show that even after closing the class of o-minimal traces under reducts and elementary equivalence we still do not cover all non-valautional structures:

Theorem 3. *Let $\mathbb{Q}^{\sqrt{2}}$ be the expansion of $(\mathbb{Q}, +)$ by the predicate $y < \sqrt{2}x$. Then $\mathbb{Q}^{\sqrt{2}}$ is non-valuational and not elementarily equivalent to a reduct of an o-minimal trace.*

Along the way we reveal a new dividing line between two types of non-valuational structures:

- *Tight* structures (of which $\mathbb{Q}^{\sqrt{2}}$ is a typical example), in which $\mathcal{M}^*$ is interpretable in $\mathcal{M}$. These are *small* (in the sense of [9]), and in that respect differ significantly from o-minimal traces.
- Non-tight structures, whose theory resembles to a much greater extent that of o-minimal traces.

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2. Preliminaries

We fix a non-valuational structure $\mathcal{M}$ and its definable completion $\bar{M}$. Recall that the elements of $\bar{M}$ are all (unique) realizations of definable cuts in $\mathcal{M}$. These will be identified here with the definable open subsets of M that are bounded above and downward closed. The set $\bar{M}$ is equipped with ordering by inclusion. The structure $\langle M, <\rangle$ is naturally embedded into $\bar{M}$ via the map $a \mapsto (-\infty, a)$, and from now on we will view M as a subset of $\bar{M}$. The topology on $\bar{M}$ and $\bar{M}^n$ are the order and the product topology, respectively. We let $\mathrm{cl}_{\bar{M}}(-), \partial_{\bar{M}}(-)$ denote the corresponding topological operations in $\bar{M}^n$. Unless otherwise stated, all definability below refers to the structure $\mathcal{M}$.

Recall that a partial function $f : M^n \to \bar{M}$ is said to be *definable* if the set $\{(x, y) \in M^{n+1} : y < f(x)\}$ is definable. Equivalently, the family of cuts $\{y \in M : y < f(x)\}$, for $x \in M^n$, is a definable family (and can be identified with a sort in $\mathcal{M}$).

We start by collecting several useful facts concerning the relationship of $\mathcal{M}$ and various structures on $\bar{M}$. We first recall the definition of a strong cell $C \subseteq M^n$ from [10] [1] The definition will be inductive in n and for the induction step we will also associate inductively to each strong cell $C \subseteq M^n$ its so-called *iterative convex hull* $\bar{C}$, $C \subseteq \bar{C} \subseteq \bar{M}^n$. Having defined C and $\bar{C}$ below, we say that an $\mathcal{M}$-definable function $f : C \to \bar{M}$ is *strongly continuous* if it extends continuously to $\bar{f} : \bar{C} \to \bar{M}$, and in addition either $f(C) \subseteq M$ or $f(C) \subseteq \bar{M} \setminus M$. We are now ready to state the definition:

Definition 2.1. A set $C \subseteq M$ is *a strong cell* if it is either a point, in which case $\bar{C} = C$, or an open convex set, in which case $\bar{C}$ is defined as the convex hull of C in $\bar{M}$.

Inductively, If $C \subseteq M^n$ is a strong cell (with the associated $\bar{C} \subseteq \bar{M}^n$) and $f, g : C \to \bar{M}$ are strongly continuous with $\bar{f}(x) < \bar{g}(x)$ for all $x \in \bar{C}$ (note the strong assumption here!) then $\Gamma_f(C)$ – the graph of f on C – and $(f, g)_C := \{(x, y) \in M^{n+1} : f(x) < y < g(y)\}$ *are strong cells.* In the first case the iterative convex hull is defined to be the graph of the extension $\bar{f} : \bar{C} \to \bar{M}$, and in the second case it is defined to be

$$\{(x, y) \in \bar{M}^{n+1} : x \in \bar{C} \;\&\; \bar{f}(x) < y < \bar{g}(x)\}.$$

Remark 2.2. (1) It is easy to verify that for each strong cell $C \subseteq M^n$ there exists a homeomorphic projection $\pi_C : C \to D \subseteq M^k$ onto k of the coordinates, $k \leq n$, whose image is an open strong cell in M^k. In this case $\dim C := k$. The coordinate functions of π_C^{-1} are strongly continuous on D.

[1] We are using Wencel's definition, in a slightly different formulation than in [6].

(2) Notice that each strong cell C is a subset of M^n that is definable in $\mathcal{M}$, and furthermore the various functions f and g in the inductive definition of C are definable in $\mathcal{M}$, even though they might take values in $\bar{M} \setminus M$. However, in general $\bar{C} \subseteq \bar{M}^n$ is not definable in $\mathcal{M}$ in any obvious sense because it might not be contained in finitely many sorts in $\mathcal{M}$.

We can now describe Wencel's canonical completion $\bar{\mathcal{M}}$, but we refine his definition so we have a better control of parameters.

Definition 2.3. Given $A \subseteq M$, we let $\bar{\mathcal{M}}_A$ be the expansion of $\bar{M}$ by all iterative convex hulls $\bar{C} \subseteq \bar{M}^n$, so that $C \subseteq M^n$ is a strong cell defined over A.

It is easy to see that the order relation $<$ is an atomic relation in $\bar{\mathcal{M}}_A$. Since $\langle M, <, +\rangle$ is divisible, [6], and M is dense in $\bar{M}$, the group operation extends uniquely to $\bar{M}$, so it is strongly continuous, and its graph C_+ is a strong cell whose iterative convex hull is the graph of a group operation on $\bar{M}$ that we still denote by $+$.

We now collect some of the main results from [11]

Fact 2.4. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational structure.*

(1) *Every A-definable set has a decomposition into finitely many strong cells, each defined over A.*
(2) *The structure $\bar{\mathcal{M}}_M$ is o-minimal.*
(3) *If $X \subseteq \bar{M}^n$ is definable in $\bar{\mathcal{M}}$ then $X \cap M^n$ is definable in $\mathcal{M}$.*

In [5], the language of $\bar{\mathcal{M}}_A$ was replaced by another one, which we find more convenient to work with.

Definition 2.5. Given $A \subseteq M$, and an A-definable set $X \subseteq M^n$ in $\mathcal{M}$, we associate to X a predicate symbol $\hat{X}$. We interpret $\hat{X}$ in $\bar{M}^n$ as the topological closure of X in $\bar{M}^n$, denoted by $\mathrm{cl}_{\bar{M}}(X)$, and let $\mathcal{M}^*_A$ be the expansion of $\bar{M}$ by all $\hat{X}$, for $X \subseteq M^n$ definable over A.

It was proved in [5] that the structures $\bar{\mathcal{M}}_M$ and and $\mathcal{M}^*_M$ have the same definable sets. We re-prove here a more precise version. We first prove:

Lemma 2.6. *If $C \subseteq M^n$ is a strong cell then $\mathrm{cl}_{\bar{M}}(C) = \mathrm{cl}_{\bar{M}}(\bar{C})$.*

Proof. Since $C \subseteq \bar{C}$ it suffices to show that $\bar{C} \subseteq \hat{C}$ for every strong cell C. We use induction on n.

If $C \subseteq M$ the claim is obvious. Now, suppose that $\bar{C} \subseteq \hat{C}$ for some strong cell C and let $f_1, f_2 < f_3$ be strongly continuous such that the range of f_1 is in M. We let $C_1 = \Gamma(f_1)_C$ and $C_2 = (f_2, f_3)_C$ be the associated strong cells, and will show that $\bar{C}_1 \subseteq \hat{C}_1$ and $\bar{C}_2 \subseteq \hat{C}_2$.

Let $(c, m) \in \bar{C} \times \bar{M}$. If $(c, m) \in \bar{C}_1$ then $\bar{f}_1(c) = m$. But then since $c \in \bar{C}$ and $\Gamma(f_1)_C$ is dense in $\Gamma(\bar{f}_1)_{\bar{C}}$ (because C is dense $\bar{C}$) then (c, m) is a limit point of f_1 and therefore $(c, m) \in \hat{C}_1$. If $(c, m) \in \bar{C}_2$ then $\bar{f}_2(c) < m < \bar{f}_3(c)$, and again since $c \in \bar{C}$ and $(f_2, f_3)_C$ is dense in $(\bar{f}_2, \bar{f}_3)_{\bar{C}}$ then (c, m) is a limit point of $(f_2, f_3)_C$ and therefore $(c, m) \in \hat{C}_2$. □

We can now prove:

Proposition 2.7. *For every $A \subseteq M$, the (o-minimal) structures $\mathcal{M}^*_A$ and $\bar{\mathcal{M}}_A$ have the same $\emptyset$-definable sets (so in particular the same definable sets).*

Proof. We first show that every atomic set in $\mathcal{M}^*_A$ is $\emptyset$-definable in $\bar{\mathcal{M}}_A$. So we take an A-definable $X \subseteq M^k$, and consider its closure $\hat{X} \subseteq \bar{M}^k$. By Fact 2.4, X can be written as the union $\bigcup_{i=1}^k C_i$ of strong cells that are definable over A in $\mathcal{M}$. By Lemma 2.6, each $\bar{C}_i$ is dense in $\mathrm{cl}_{\bar{M}}(C_i)$. It follows that $\mathrm{cl}_{\bar{M}}(X) = \bigcup_{i=1}^k \mathrm{cl}_{\bar{M}}(\bar{C}_i)$. Since each $\bar{C}_i$ is $\emptyset$-definable in $\bar{\mathcal{M}}_A$, and the closure operation is itself definable, it follows that $\mathrm{cl}_{\bar{M}}(X)$ is $\emptyset$-definable in $\bar{\mathcal{M}}_A$.

For the other inclusion, we need to see that for every strong cell $C \subseteq M^n$ that is definable over A, the set $\bar{C}$ is $\emptyset$-definable in $\mathcal{M}^*_A$. This is done by induction on n.

For 0-cells in $\mathcal{M}$ this is clear. If $C \subseteq M$ is a 1-cell then $\bar{C}$ is an open interval (a, b) in $\bar{M}$. The interval $[a, b]$ is $\emptyset$-definable in $\mathcal{M}^*_A$, hence so is $\bar{C}$. So we now assume that we have proved the result for all strong cells in M^n and we prove it for strong-cells in M^{n+1}. Let $C \subseteq M^n$ be a strong cell defined over A. Let $f : C \to M$ be a strongly continuous function definable in $\mathcal{M}$ over A, and let Y be Γ_f, the graph of f. Then $\bar{Y} := \{(x, \bar{f}(x)) : x \in \bar{C}\}$. We have to show that $\bar{Y}$ is $\emptyset$-definable in $\mathcal{M}^*_A$. As $\bar{f}$ is continuous we get that $\bar{Y} = \mathrm{cl}_{\bar{M}}(\Gamma_f) \cap (\bar{C} \times \bar{M})$, which is $\emptyset$-definable in $\mathcal{M}^*_A$ by the inductive hypothesis.

Now let $f, g : C \to \bar{M}$ be A-definable strongly continuous functions in $\mathcal{M}$, with $f < g$ (unlike the above, we cannot assume here that they take values in M). We have to show that the iterative convex hull of $Y := (f, g)_C$ is $\emptyset$-definable in $\mathcal{M}^*_A$. By definition,

$$\bar{Y} = \{(x, y) : x \in \bar{C}, \bar{f}(x) < y < \bar{g}(x)\}.$$

Since, by induction $\bar{C}$ is $\emptyset$-definable in $\mathcal{M}^*_A$, it will suffice to show that $\bar{f}$ (and similarly $\bar{g}$) is $\emptyset$-definable in $\mathcal{M}^*_A$. If f is the constant function $-\infty$, then there is nothing to prove. So we assume this is not the case. By definition, the set

$$F := \{(x, y) : x \in C, y < f(x)\}$$

is A-definable in $\mathcal{M}$. For every $c \in \bar{C}$ let

$$s(c) := \sup\{y \in \bar{M} : (c, y) \in \mathrm{cl}_{\bar{M}}(F)\}.$$

Since f is strongly continuous, $s(c)$ is well defined, and by definition it coincides with f on C. Since C is dense in $\bar{C}$ and $\bar{f}$ is the unique continuous extension of f to $\bar{C}$, necessarily $s = \bar{f}$, and as s is $\emptyset$-definable in $\mathcal{M}^*_A$, we are done. □

From now on we can use interchangeably the structures $\mathcal{M}^*_A$ and $\bar{\mathcal{M}}_A$. Notice however, that the language of $\bar{\mathcal{M}}_A$ depends on the specific structure $\mathcal{M}$, thus for different $\mathcal{M}$ and $\mathcal{N}$, even if elementarily equivalent, the structures $\bar{\mathcal{M}}_M$ and $\bar{\mathcal{N}}_N$ are of different signature. One of the initial goals of this work was to obtain a uniform signature by showing that the definable sets in $\bar{\mathcal{M}}_\emptyset$ ad $\bar{\mathcal{M}}_M$ are the same. We need the following observations

Proposition 2.8. (1) *Every $\emptyset$-definable set in $\mathcal{M}$ can be written as a boolean combination of $\emptyset$-definable sets each of which is the closure of an open $\emptyset$-definable set. In particular, this is true if $\mathcal{M}$ is o-minimal.*
(2) *The o-minimal structure $\mathcal{M}^*_M$ eliminates quantifiers. Moreover, it is sufficient to take as atomic relations all $\mathrm{cl}_{\bar{M}}(X)$ with $X \subseteq M^n$ an open definable set.*

Proof. (1) We first prove the result for an arbitrary definable open set $X \subseteq M^n$. Note that $X = \mathrm{cl}(X) \setminus \partial(X)$ (here $\partial(X)$ is the boundary of X), and then that

$$\partial(X) = \mathrm{cl}(X) \cap \mathrm{cl}(M^n \setminus \mathrm{cl}(X)).$$

The set on the right is of the desired form, so we are done.

For an arbitrary definable $X \subseteq M^n$, we apply strong cell decomposition, so we may assume that X is a cell. Hence, X is either a point or the graph of a definable map f from an open cell $C \subseteq M^{n-k}$ into M^k (the $n - k$ coordinates need not be the first ones), and each of the coordinate functions of f are strongly continuous.

Thus it is sufficient to show that the graph of each strongly continuous $f_i : C \to M$ is definable in the desired form. By the continuity of f_i, such a graph can be written as the complement in $C \times M$ of the open set:

$$\{(x, y) \in C \times M : y > f_i(x)\} \cup \{(x, y) \in C \times M : y < f_i(x)\}.$$

Since each of the open sets can be defined in the required form, so is the graph of f_i, and hence so is X. □

For (2), we first apply (1) to the o-minimal structure $\mathcal{M}^*_M$ and reduce the problem to definable sets $\hat{X} \subseteq \bar{M}^n$, which are the closure of an open definable set $U \subseteq \bar{M}^n$. Since M^n is dense in $\bar{M}^n$, $\mathrm{cl}_{\bar{M}}(U) = \mathrm{cl}_{\bar{M}}(U \cap M^n)$. By fact 2.4, the set $U \cap \bar{M}^n$ is definable in $\mathcal{M}$ (possibly over parameters). We now apply (1). □

In the text the first part of the above proposition will be applied, mostly, when $\mathcal{M}$ is, in fact, o-minimal.

Lemma 2.9. *Let $C \subseteq M^{k+n}$ be a strong cell, $a \in \pi(C)$, where π is the projection onto the first k-coordinate. Let $C_a = \{x \in M^n : (a, x) \in C\}$. Then*
(1) *C_a is a strong cell.*
(2) *$(\bar{C})_a = \overline{C_a}$.*

Proof. It is sufficient to prove the result for $k = 1$ (and then proceed by induction). This is straightforward from the definition of a strong cell. □

Theorem 2.10. *For every $A \subseteq M$, the structures $\bar{\mathcal{M}}_M$ and $\bar{\mathcal{M}}_A$ have the same definable sets.*

Proof. Absorbing A to the language we, at this stage, assume that $A = \emptyset$. We first claim that for every $n \in \mathbb{N}$, we have

$$\{Y \cap M^n : Y \subseteq \bar{M}^n \text{ definable in } \bar{\mathcal{M}}_\emptyset\} = \{Y \cap M^n : Y \subseteq \bar{M}^n \text{ definable in } \bar{\mathcal{M}}_M\}. \quad (2)$$

Since $\bar{\mathcal{M}}_\emptyset$ is a reduct of $\bar{\mathcal{M}}_M$ it is sufficient to prove the right-to-left inclusion. We first show: For every $\emptyset$-definable $X \subseteq M^n$, there exists a $\emptyset$-definable $Y \subseteq \bar{M}^n$ in $\bar{\mathcal{M}}_\emptyset$ such that $Y \cap M^n = X$. Indeed, X has a decomposition into $\emptyset$-definable strong cells (see Fact 2.4), and for each $\emptyset$-definable strong cell C_i we have $\bar{C}_i \cap M^n = C_i$, so $Y = \bigcup_i \bar{C}_i$ is the desired set.

Now, let $Z \subseteq M^n$ be definable in $\bar{\mathcal{M}}_M$. By Fact 2.4, $Z \cap M^n$ is definable in $\mathcal{M}$, possibly over parameters. Hence, it is of the form X_a, for some $X \subseteq M^{n+k}$ which is $\emptyset$-definable in $\mathcal{M}$ and $a \in M^k$. By what we just shown, there is $Y \subseteq \bar{M}^k$ which is $\emptyset$-definable in $\bar{\mathcal{M}}_\emptyset$, such that $X = Y \cap M^{n+k}$. Hence,

$$Z = (Y \cap M^{n+k})_a = Y_a \cap M^n,$$

and Y_a is definable in $\bar{\mathcal{M}}_\emptyset$. This ends the proof of (2).

We now make the following general observation:

Lemma 2.11. *Let $\langle N, <\rangle$ be a densely ordered set, with $M \subseteq N$ a dense subset. Assume that $\mathcal{N}_1$, $\mathcal{N}_2$ are two o-minimal expansions of $\langle N, <\rangle$ with the property that for every $n \in \mathbb{N}$, we have*

(3) $$\{Y \cap M^n : Y \subseteq \bar{M}^n \textit{ definable in } \mathcal{N}_1\} = \{Y \cap M^n : Y \subseteq \bar{M}^n \textit{ definable in } \mathcal{N}_2\}.$$

Then $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same definable sets.

Proof. It easily follows from the assumptions that we have

(4) $$\{Y \cap M^n : Y \subseteq \bar{M}^n \text{ open definable in } \mathcal{N}_1\} = \{Y \cap M^n : Y \subseteq \bar{M}^n \text{ open definable in } \mathcal{N}_2\}.$$

By Proposition 2.8 (1), it is enough to know that for every open $U \subseteq N^n$, the set $\mathrm{cl}(U)$ is definable in $\mathcal{N}_1$ if and only if it is definable in $\mathcal{N}_2$. However, since M is dense in N, it is enough to consider sets of the form $\mathrm{cl}(U \cap M^n)$. By (4), both collections of sets of the form $U \cap M^n$, where U is definable in either $\mathcal{N}_1$ or in $\mathcal{N}_2$, are the same.. □

In order to prove Theorem 2.10, we apply Lemma 2.11 to the structures $\bar{\mathcal{M}}_\emptyset$ and $\bar{\mathcal{M}}_M$ using (2).

□

3. The structure $\mathcal{M}_A$ and elementary extensions

Again, we let $\mathcal{M}$ be a fixed non-valuational structure . From now on we shall work with $\mathcal{M}^*_A$ rather than $\bar{\mathcal{M}}_A$.

3.1. **The canonical completion and elementary extensions.** Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. Every definable cut C in $\mathcal{M}$ has a natural realization $C(\mathcal{N})$ in $\mathcal{N}$ and so $\bar{M}$ can be embedded into $\bar{N}$. Under this embedding, if $n \in \bar{N}$ is the supremum of a cut in N which is definable over some $A \subseteq M$ then n is already in $\bar{M}$. We have:

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\iota} & \bar{\mathcal{N}} \\ \curlyvee & & \uparrow \\ \mathcal{M} & \xrightarrow{\iota} & \bar{\mathcal{M}} \end{array}$$

Where ι is the natural embedding of $(M, <)$ in $(\bar{M}, <)$. We now fix an arbitrary $A \subseteq M$ and consider the structures $\mathcal{M}^*_A$ and $\mathcal{N}^*_A$. Both structures are in the language $\mathcal{L}^*_A$, and we claim that $\mathcal{M}^*_A$ is a substructure of $\mathcal{N}^*_A$: Indeed, first note that for *a fixed* $x \in \bar{M}^n$, and $\epsilon > 0$ in M, the set $B(x, \epsilon) \cap M^n = \{y \in M^n : |x - y| < \epsilon\}$ is definable in $\mathcal{M}$ and moreover, it is uniformly definable as ϵ varies in $M_{>0}$ (x still fixed). It easily follows that for $x \in \bar{M}^n$, being in the closure of a definable $X \subseteq M^n$ is a first order property. Namely, for $x \in \bar{M}^n$,

$$x \in \mathrm{cl}_{\bar{M}}(X(M)) \Leftrightarrow x \in \mathrm{cl}_{\bar{N}}(X(N)).$$

Said differently, $\hat{X}(N) \cap \bar{M}^n = \hat{X}(M)$, so $\mathcal{M}^*_A$ is a substructure of $\mathcal{N}^*_A$.

Our goal is to show that $\mathcal{M}^*_A$ is in fact an elementary substructure of $\mathcal{N}^*_A$.We do that in several steps.

Lemma 3.1. *Assume that $A \subseteq M$ and that $\mathcal{M}$ is $|A|^+$-saturated. If $Y \subseteq \bar{M}^n$ is $\emptyset$-definable in $\mathcal{M}^*_A$ then $Y \cap M^n$ is A-definable in $\mathcal{M}$.*

Proof. By fact 2.4, $Y \cap M^n$ is definable in $\mathcal{M}$. By the saturation assumption it is enough to show that any automorphism of $\mathcal{M}$ which fixes A point-wise leaves $Y \cap M^n$ invariant. Let $\alpha : M \to M$ be such an automorphism. We claim that α has a (unique) extension to a bijection $\bar{\alpha} : \bar{M} \to \bar{M}$ which is an automorphism of $\mathcal{M}^*_A$. Because α is an automorphism of $\mathcal{M}$ it sends definable cuts to definable cuts so extends naturally to $\bar{\alpha} : \bar{M} \to \bar{M}$. The map $\bar{\alpha}$ is an order preserving bijection so in particular continuous on $\bar{M}$. To see that $\bar{\alpha}$ is an automorphism of $\mathcal{M}^*_A$, let $X \subseteq M^n$ be A-definable and consider its closure $\hat{X}$. Since $\alpha(X) = X$, continuity implies that $\bar{\alpha}(\hat{X}) = \hat{X}$, thus $\bar{\alpha}$ is an automorphism of $\mathcal{M}^*_A$.

Since Y was $\emptyset$-definable in $\mathcal{M}^*_A$ it is left invariant under $\bar{\alpha}$, and because $\bar{\alpha}(M) = M$, we have

$$\alpha(Y \cap M^n) = \bar{\alpha}(Y \cap M^n) = \bar{\alpha}(Y) \cap \bar{\alpha}(M^n) = Y \cap M^n.$$

□

Lemma 3.2. *For $A \subseteq M$ arbitrary, if $\mathcal{M} \prec \mathcal{N}$ then $\mathcal{M}^*_A \prec \mathcal{N}^*_A$.*

Proof. First note that we may assume that $\mathcal{N}$ is sufficiently saturated. Indeed, we may consider $\mathcal{N}' \succ \mathcal{N}$ which is saturated enough. The above would then imply that $\mathcal{M}^*_A \prec (\mathcal{N}'_A)^*$ and $\mathcal{N}^*_A \prec (\mathcal{N}'_A)^*$, from which it follows that $\mathcal{M}^*_A \prec \mathcal{N}^*_A$.

By The Tarski-Vaught Criterion, it is enough to prove, for every nonempty $Y \subseteq \bar{N}$ which is definable in $\mathcal{N}^*_A$ over $\bar{M}$, that $Y \cap \bar{M} \neq \emptyset$.

Since $\mathcal{N}^*_A$ is an o-minimal expansion of a group, Y contains some element $b \in \mathrm{dcl}_{\mathcal{N}^*_A}(\bar{M})$. So, there exists a finite tuple $a = (a_1, \dots, a_r)$ from $\bar{M}$, such that $b \in \mathrm{dcl}_{\mathcal{N}^*_A}(a)$. Each a_i realizes a cut in M, definable in $\mathcal{M}$ over some finitely many parameters. Thus there is a finite $F \subseteq M$ such that each a_i realizes a cut definable over F. If we now let $A' = A \cup F \subseteq M \subseteq N$ then clearly every element in A' is $\emptyset$-definable in $\mathcal{N}^*_{A'}$, hence b is in $dcl_{\mathcal{N}^*_{A'}}(\emptyset)$ so the set $(-\infty, b)$ is $\emptyset$-definable in $\mathcal{N}^*_{A'}$. Since $\mathcal{N}$ is sufficiently saturated it follows from Lemma 3.1 that $(-\infty, b) \cap N$ is A'-definable in $\mathcal{N}$.

Since $\mathcal{M} \prec \mathcal{N}$ and $A' \subseteq M$ it follows, as we already noted above, that $b \in \bar{M}$, so $X \cap \bar{M} \neq \emptyset$. Thus $\mathcal{M}^*_A \prec \mathcal{N}^*_A$. □

Note: It only makes sense to compare $\mathcal{M}^*_A$ and $\mathcal{N}^*_A$ for $A \subseteq M$, since otherwise the two structures do not have a common language.

Finally, we can now prove:

Theorem 3.3. *For $A \subseteq M$ (with no saturation assumption), assume that $X \subseteq \bar{M}^n$ is $\emptyset$-definable in the structure $\mathcal{M}^*_A$. Then $X \cap M^n$ is A-definable in $\mathcal{M}$. In particular, if $f : \bar{M}^n \to \bar{M}$ is $\emptyset$-definable in $\mathcal{M}^*_A$ then $f \restriction M^n : M^n \to \bar{M}$ is A-definable in $\mathcal{M}$.*

Proof. We consider an elementary extension $\mathcal{N}$ of $\mathcal{M}$ that is $|A|^+$-saturated. By Lemma 3.2, we have $\mathcal{M}^*_A \prec \mathcal{N}^*_A$ and by Lemma 3.1, the set $Y = X(\bar{N}) \cap N^n$ is definable in $\mathcal{N}$ over A. Since $\mathcal{M} \prec \mathcal{N}$ we can conclude that $Y \cap M^n = X(\bar{N}) \cap M^n$ is also definable over A in $\mathcal{M}$. It is left to see that this last set equals $X \cap M^n$. Because $\mathcal{M}^*_A \prec \mathcal{N}^*_A$ we have $X(\bar{N}) \cap \bar{M}^n = X$, and therefore

$$Y \cap M^n = X(\bar{N}) \cap M^n = X \cap M^n.$$

For the second clause, just note that the set $\{x \in M^n : x < f(x)\}$ is the intersection of a $\emptyset$-definable subset of $\bar{M}^n$ with M^n. □

We now return to Proposition 2.8 and Theorem 2.10 and prove finer results:

Proposition 3.4. *For any $A \subseteq M$,*

(1) *The structure $\mathcal{M}^*_A$ eliminates quantifiers. In fact every $\emptyset$-definable set is a boolean combination of sets of the form $\mathrm{cl}_{\bar{M}}(X)$ for $X \subseteq M^n$ open and definable in $\mathcal{M}$ over A.*
(2) *If $X \subseteq \bar{M}^n$ is $\emptyset$-definable in $\mathcal{M}^*_A$ then it is A-definable in $\mathcal{M}^*_\emptyset$.*

Proof. (1) We may repeat the short argument in the proof of 2.8 with the additional data given by Theorem 3.3, that whenever $X \subseteq \bar{M}^n$ is $\emptyset$-definable in $\mathcal{M}^*_A$, the set $X \cap M^n$ is A-definable in $\mathcal{M}$. For (2), assume that Z is $\emptyset$-definable in M^*_A. By (1), Z is a boolean combination of atomic sets (with no extra parameters), so it is sufficient to prove that each atomic such set Z is A-definable in $\mathcal{M}^*_\emptyset$. By the first paragraph of the proof of Proposition 2.7 $Z = \bigcup_{i=1}^k \mathrm{cl}_{\bar{M}}(\bar{C}_i)$ for some A-definable strong cells $C_i \subseteq M^n$. So $C_i = (D_i)_a$ for some $\emptyset$-definable set D_i nd $a \subseteq A$. By strong cell decomposition, each D_i is itself a finite union of $\emptyset$-definable strong cells, so we may write each C_i as a union of the form $\bigcup_j (D_{i,j})_a$, where each $D_{i,j}$ is a $\emptyset$-definable strong cell.

By Lemma 2.9 we know that $\overline{(D_{i,j})_a} = (\overline{D_{i,j}})_a$ for every j. The right-hand side of this equation is A-definable in $\mathcal{M}^*_\emptyset$, and hence so is its $\bar{M}$-closure. Therefore the closure of each C_i is a finite union of sets that are A-definable in $\mathcal{M}^*_\emptyset$. The conclusion follows. □

Since any two elementarily equivalent structures have a common elementary extension, we can also conclude from Lemma 3.2:

Corollary 3.5. *If $\mathcal{M}$ is non-valuational and $\mathcal{N} \equiv \mathcal{M}$ then $\bar{\mathcal{M}}_\emptyset \equiv \bar{\mathcal{N}}_\emptyset$ (both are $\bar{\mathcal{L}}_\emptyset$-structures), and $\mathcal{M}^*_\emptyset \equiv \mathcal{N}^*_\emptyset$ (as $\mathcal{L}^*_\emptyset$-structures)*

Finally, we shall be using the following technical lemma:

Lemma 3.6. *For every $A \subseteq M$, $\mathrm{dcl}_{\mathcal{M}^*_\emptyset}(A) \cap M = \mathrm{dcl}_{\mathcal{M}}(A)$.*

Proof. Assume that $a \in dcl_{\mathcal{M}^*_\emptyset}(A) \cap M$. Then it follows that $a \in dcl_{\mathcal{M}^*_A}(\emptyset)$ (since each element of A is $\emptyset$-definable in $\mathcal{M}^*_A$). Hence, the interval $(-\infty, a) \subseteq \bar{M}$ is $\emptyset$-definable in $\mathcal{M}^*_A$, so by Theorem 3.3, the intersection of $(-\infty, a)$ with M is A-definable in $\mathcal{M}$. Because $a \in M$, we have $a \in dcl_{\mathcal{M}}(A)$.

For the converse, assume that $a \in dcl_{\mathcal{M}}(A)$ (so in particular in M). Thus, the interval $(-\infty, a)$ is definable in $\mathcal{M}$ over A and its iterative convex hull, the interval $\overline{(\infty, a)} \subseteq \bar{M}$, is $\emptyset$-definable in $\mathcal{M}_A$. By Proposition 2.8(2), this interval is A-definable in $\mathcal{M}^*_\emptyset$ so $a \in \mathrm{dcl}_{\mathcal{M}^*}(A) \cap M$. □

4. Tight weakly o-minimal structures

As was pointed out before, the set $\bar{M}$ can be viewed as a union of sorts in $\mathcal{M}$, where each sort corresponds to a $\emptyset$-definable family of cuts in $\mathcal{M}$. In general, there might be infinitely many such sorts, but in some cases there are only finitely many such sorts.

4.1. **Definition and basic properties.**

Definition 4.1. A non-valuational structure $\mathcal{M}$ is *tight* if there are finitely many $\emptyset$-definable families of cuts in $\mathcal{M}$ such that every definable cut belongs to one of them.

Clearly, if $\mathcal{M}$ is an o-minimal structure then it is (trivially) non-valuational and tight, since the family of definable cuts is just all intervals of the form $(-\infty, a)$, as a varies in M.

It immediately follows that if $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{M}$ is tight if and only if $\mathcal{N}$ is tight. Thus, we may use the term "tight" for T as well.

Proposition 4.2. *The structure $\mathcal{M}$ is tight if and only if there are finitely many $\emptyset$-definable functions $f_i : M^{n_i} \to \bar{M}$, $i = 1, \ldots, k$, such that $\bar{M} \subseteq \bigcup_{i=1}^{k} Im(f_i)$.*

In particular, $\mathcal{M}$ is tight then the structure $\mathcal{M}^*$ is interpretable in $\mathcal{M}$ without parameters.

Proof. The first clause is easy to verify. For the second clause, note first that the universe of $\bar{M}$ is a quotient of some M^n by a definable set, and furthermore the embedding of M in this quotient (i.e. the family of cuts $\{C_x : x \in M\}$, where $C_x = \{y < x\}$) is definable in $\mathcal{M}$. It is easy to see that the ordering on $\bar{M}$ is definable in $\mathcal{M}$ and hence $\mathrm{cl}_{\bar{M}}(X)$ is definable in $\mathcal{M}$ for every $\mathcal{M}$-definable $X \subseteq M^n$. □

Remark 4.3. The above proof shows, in fact, that the pair $(\mathcal{M}^*, \mathcal{M})$ is bi-interpretable with $\mathcal{M}$, i.e., not only is $\mathcal{M}^*$ interpretable in $\mathcal{M}$, but so is the natural embedding of M in $\bar{M}$.

4.2. **An example of a tight structure.**

We shall now see that there are examples of tight structures which are not o-minimal.

Let $\mathbb{Q}_{vs} = \langle \mathbb{Q}, <, +, 1, \{\lambda_q\}_{q \in \mathbb{Q}} \rangle$ denote the group of rational numbers, viewed as an ordered vector space over itself, with a function symbol for every rational scalar. Let $\mathbb{Q}^{\sqrt{2}}$ be the expansion of $\mathbb{Q}_{vs}$ by the relation

$$P_{\sqrt{2}} = \{(x, y) \in \mathbb{Q}^2 : y < \sqrt{2}x\}.$$

We denote the langauge by $\mathcal{L}_{\sqrt{2}}$. (In [4, Section 3] a similar expansion of $\mathbb{Q}_{vs}$ by the predicate P_π was investigated.)

The idea is to eventually identify $P_{\sqrt{2}}$ with a map $x \mapsto \sqrt{2}x$ from the structure $\mathbb{Q}^{\sqrt{2}}$ into its canonical completion. Our goal is to show that $\mathrm{Th}(\mathbb{Q}^{\sqrt{2}})$ is axiomatised by the following theory T:

(1) The ordered $\mathbb{Q}$-vector space axioms.
(2) An axiom expressing the fact that $P_{\sqrt{2}}$ is "linear":
$$(\forall x_1, y_1, x_2, y_2)\,(((x_1, y_1) \in P_{\sqrt{2}} \wedge (x_2, y_2) \in P_{\sqrt{2}}) \to (x_1 + x_2, y_1 + y_2) \in P_{\sqrt{2}}).$$
(3) (Ensuring that we define the positive $\sqrt{2}$): $(\exists x, y)((x, y) \in P_{\sqrt{2}} \wedge x > 0 \wedge y > 0)$
(4) For all $r \in \mathbb{Q}$, such that $r < \sqrt{2}$, we have $\forall x\,(x > 0 \to (x, rx) \in P_{\sqrt{2}})$, and for all $r \in \mathbb{Q}$ such that $r > \sqrt{2}$, we have $\forall x\,(x > 0 \to (x, rx) \notin P_{\sqrt{2}})$.
(5) For all $x \neq 0$, the set
$$\{y : (x, y) \in P_{\sqrt{2}}\}$$
is closed downwards, and has no supremum. Furthermore,
$$\mathrm{Inf}\,\{y_2 - y_1 : (x, y_1) \in P_{\sqrt{2}}\,\&\,(x, y_2) \notin P_{\sqrt{2}}\} = 0.$$

(6) An axiom expressing the fact that the composition of $x \mapsto \sqrt{2}x$ with itself yields the map $x \mapsto 2x$:
$$\forall (x, y > 0)\left([(\exists z > 0) P_{\sqrt{2}}(x, z) \wedge P_{\sqrt{2}}(z, y)] \iff y < 2x\right).$$
(7) The quantifier-free theory of $\mathbb{Q}^{\sqrt{2}}$.

Clearly, $\mathbb{Q}^{\sqrt{2}}$ is a model of T.

For simplicity we write $F = \mathbb{Q}(\sqrt{2})$. Before we prove quantifier elimination we note that if $\mathcal{M}$ is a model of T then we may consider the associated F-vector space $V = F \otimes_{\mathbb{Q}} M$. If we identify M with the $\mathbb{Q}$-subspace $1 \otimes M$, then each element of V can be written uniquely as $x + \sqrt{2}y$ for $x, y \in M$. We can now endow V with an ordering by declaring $x + \sqrt{2}y > 0$ when $(y, -x) \in P_{\sqrt{2}}$. Indeed, the above axioms imply that this is a linear ordering of the vector space V, compatible with the ordering of F.

The definition of the ordering and Axiom (3) allows us to conclude:

Claim 4.4. (1) *For $x, y \in M$, we have $(x, y) \in P_{\sqrt{2}} \Leftrightarrow$ if and only if $y < \sqrt{2}x$ in V.*
(2) *M is dense in V.*

We can now endow V with an $\mathcal{L}_{\sqrt{2}}$-structure, by interpreting $P_{\sqrt{2}}$ as we did over $\mathbb{Q}$. Clause (1) above then implies that $\mathcal{M}$ is a substructure of V as an $\mathcal{L}_{\sqrt{2}}$-structures.

The following lemma is similar to [4, Proposition 3.3]:

Lemma 4.5. *The theory T is complete and has quantifier elimination.*

Proof. Let $\mathcal{Q}_1, \mathcal{Q}_2 \models T$ be κ-saturated models of the same cardinality. In order to prove quantifier elimination it suffices to prove (see for example [7, Corollary 3.1.6]):

If A is a substructure of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of cardinality smaller than κ, then for every $a_1 \in Q_1$ there is $a_2 \in Q_2$ such that a_1 and a_2 have the same quantifier-free type over A.

As above, consider the ordered F-vector spaces $\mathcal{G}_i := F \otimes_{\mathbb{Q}} \mathcal{Q}_i$. Since Q_i is dense in G_i, and $\mathcal{G}_i$ is o-minimal, the saturation of $\mathcal{Q}_i$ implies that $\mathcal{G}_i$ is also κ-saturated. Let B_i be the F-span of A inside $\mathcal{G}_i$. Then B_1 and B_2 are isomorphic-over-A ordered vector spaces (both isomorphic to $A + \sqrt{2}A$, with the same ordering). Thus we may write $B = B_1 = B_2$

Let $p(x) := \mathrm{tp}_{\mathcal{G}_1}(a_1/B)$. We may assume that $a_1 \notin A$ and hence $a_1 \notin B$ (note that $B \cap Q_1 = A$). By the completeness of the theory of ordered F-vector spaces and saturation, we can find $a_2 \in \mathcal{G}_2$ such that $a_2 \models p(x)$. In fact, because $\mathcal{G}_2$ is κ-saturated and p is non-algebraic there is more than one such a_2, so since Q_2 is dense in G_2, we can find such an a_2 inside Q_2.

Finally, since each $\mathcal{Q}_i$ is a substructure of $\mathcal{G}_i$, and $a_1, a_2 \models p$, it follows that the quantifier-free types of a_1 and a_2 over A, in the structures $\mathcal{Q}_1$, $\mathcal{Q}_2$, respectively, are the same. This completes the proof of quantifier elimination.

To see that T is complete we just notice that every model of T contains the structure $\mathbb{Q}^{\sqrt{2}}$, which is itself a model of T. □

Corollary 4.6. *The theory T is a tight weakly o-minimal non-valuational theory and T^* is the theory of ordered $\mathbb{Q}^{\sqrt{2}}$-ordered vector spaces (in the language $\mathcal{L}_{\sqrt{2}}$).*

Proof. The atomic subsets of Q, the universe of any $\mathcal{Q} \models T$, are rays with or without endpoints. By quantifier elimination the definable subsets of Q are in the boolean algebra generated by those, proving the weak o-minimality. The same argument also shows that the only definable cuts are non-valuational, because so are the atomic cuts.

By the proof of lemma 4.5 and the preceding discussion, each model $\mathcal{Q}$ of T is a dense substructure of the o-minimal structure $V = F \otimes_{\mathbb{Q}} Q$. It is easy to verify that the intersection with Q of every ray $(-\infty, a)$ in V is definable in $\mathcal{Q}$, and hence every element of V realizes a definable cut in $\mathcal{Q}$. Conversely, by quantifier elimination, the definable cuts in any model $\mathcal{Q}$ of T are of the form $x_1 + r\sqrt{2}x_2$ for $r \in \mathbb{Q}$ and $x_1, x_2 \in Q$, so they are realized in V. It follows that V is the canonical completion of $\mathcal{Q}$, and its theory, in the language $\mathcal{L}_{\sqrt{2}}$, is that of an ordered F-vector space.

To see that T is tight we note that each definable cut in $\mathcal{Q}$ can also be written as $x_1 + \sqrt{2}x_2$, for $x_1, x_2 \in Q$, and that this is a definable family in T. □

Note that the above construction worked because of the algebraicity of $\sqrt{2}$. If we consider $\mathbb{Q}^t$, the expansion of $\mathbb{Q}_{vs}$ by $x \mapsto tx$ where t realizes a cut defining a real transcendental number we would not obtain a tight structure. See the example $\mathbb{Q}^{\pi}_{vs}$ in [4].

We now prove:

Theorem 4.7. *The structure $\mathbb{Q}^{\sqrt{2}}$ is not elementarily equivalent to a reduct of an o-minimal trace.*

Proof. Assume towards a contradiction that there is a dense pair $(\mathcal{R}, \mathcal{Q})$ of o-minimal expansions of groups such that $\mathbb{Q}^{\sqrt{2}}$ is elementarily equivalent to a reduct of the trace which this pair induces on a structure $\mathcal{Q}$. By that we mean that there is some expansion $\hat{\mathcal{Q}}$ of $\langle \mathcal{Q}, <, +\rangle$ satisfying T, such that every definable set in $\hat{\mathcal{Q}}$ is definable in the dense pair $(\mathcal{R}, \mathcal{Q})$. While some of these sets are already definable in the o-minimal structure $\mathcal{Q}$ others may be the intersection with Q^n of subsets of R^n that are definable in $\mathcal{R}$ over parameters which are not in $\mathcal{Q}$. The order relation $<$ and the group operation $+$ are assumed to be definable in $\mathcal{Q}$.

Let us consider the predicate $P\sqrt{2}(\hat{\mathcal{Q}})$. It is a definable set in $(\mathcal{R}, \mathcal{Q})$, hence by [9, Theorem2], there is a definable $Y_{\sqrt{2}} \subseteq R^2$ in the o-minimal structure $\mathcal{R}$ such that $Y_{\sqrt{2}} \cap \mathcal{Q} = \mathcal{P}\sqrt{2}(\hat{\mathcal{Q}})$. Because $\mathcal{Q}$ is dense in R (the universe of the o-minimal structure $\mathcal{R}$), it easily follows that for every $x \in Q$, there is $y(x) \in R$ such that

$$y(x) = sup\{y \in \mathcal{Q} : (x, y) \in Y_{\sqrt{2}}\}.$$

By taking the closure of the graph of $y(x)$ we obtain an $\mathcal{R}$-definable function, which we will denote by $\lambda_{\sqrt{2}} : R \to R$, which gives $y(x)$ for every $x \in \mathcal{Q}$. It is not hard to see that $\lambda_{\sqrt{2}}$ is a definable automorphism of $\langle R, +\rangle$ satisfying $\lambda_{\sqrt{2}} \circ \lambda_{\sqrt{2}}(x) = 2x$.

We now consider two cases. If the function $\lambda_{\sqrt{2}}$ is $\emptyset$-definable in $\mathcal{R}$ then it comes from a definable function in the o-minimal structure $\mathcal{Q}$, and in particular, for every $x \in \mathcal{Q}$, the set $\{y \in \mathcal{Q} : (x, y) \in P\sqrt{2}\}$ has a supremum in $\mathcal{Q}$. This contradicts the axioms of T.

On the other hand, if $\lambda_{\sqrt{2}}$ is not $\emptyset$-definable then by [8], one can define in the o-minimal structure $\mathcal{R}$ a multiplication function $\cdot$ on R^2, making $\langle R, <, +, \cdot\rangle$ a real closed field, call it K. A-priori the multiplication function might not be $\emptyset$-definable but in that case there is a $\emptyset$-definable family of such multiplications all of which expand $\langle R, +\rangle$ to a real closed field. By definable choice we may find one such multiplication function that is $\emptyset$-definable.

Since $\lambda_{\sqrt{2}}$ is an $\mathcal{R}$-definable automorphism of the additive group of K it must be of the form $x \mapsto c \cdot x$ for some scalar $c \in K$. Because $\lambda_{\sqrt{2}} \circ \lambda_{\sqrt{2}}(x) = 2x$, and because $\lambda_{\sqrt{2}}$ takes positive values on $x > 0$, the scalar c is necessarily $\sqrt{2}$ (in the sense of K). In particular, $\lambda_{\sqrt{2}}$ is $\emptyset$-definable in $\mathcal{R}$, yielding a contradiction as before. □

5. THE THEORY OF $(\mathcal{M}^*, \mathcal{M})$

From now on, given a complete non-valuational theory T we will denote by T^* the theory of the associated o-minimal completion, in the language $\mathcal{L}^*_\emptyset$ (by Corollary 3.5, the theory T indeed determines T^*). We write $\bar{\mathcal{M}}$ and $\mathcal{M}^*$, for the structure $\bar{\mathcal{M}}_\emptyset$, and $\mathcal{M}^*_\emptyset$, respectively.

While $\mathcal{M}$ and $\mathcal{M}^*_\emptyset$ initially have different signatures it will be convenient to treat them in the same langauge. We thus modify the language of $\mathcal{M}$.

Lemma 5.1. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational structure. Let $\mathcal{M}_0$ be the reduct of $\mathcal{M}$ generated by all $\emptyset$-definable closed sets. Then every $\emptyset$-definable set in $\mathcal{M}$ is $\emptyset$-definable in $\mathcal{M}_0$. In particular, $\mathcal{M}$ and $\mathcal{M}_0$ have the same $\emptyset$-definable sets.*

Proof. This follows from the proof of Proposition 2.8. □

So from now on we will assume that $\mathcal{M}$ is given in the signature consisting of a function symbol for $+$, the ordering $<$, and a predicate for each $\emptyset$-definable closed set in M^n. We let $\mathcal{L}$ be the associated language, so we may use the same language for $\mathcal{M}^*$. By Proposition 4.5, the structure $\mathcal{M}^*$ eliminates quantifiers.

We let $\mathcal{L}^P = \mathcal{L} \cup \{P\}$, where P is a unary predicate. We consider the $\mathcal{L}^P$-structure

$$\mathcal{M}^P = \langle \mathcal{M}^*, \mathcal{M}\rangle,$$

where the interpretation of P is M. As we will see, the theory of $\mathcal{M}^P$ depends only on T. We propose the following axiomatization for this theory:

Let T^d be the $\mathcal{L}^P$-language axiomatized as follows (we write $(\mathcal{M}', \mathcal{M})$ for models of T^d),

(1) $\mathcal{M} \models T$, $\mathcal{M}' \models T^*$.
(2) M dense in M'.
(3) Every definable cut in $\mathcal{M}$ has a supremum in M'.
(4) (when T is tight) Every element of $\mathcal{M}'$ realizes a definable cut in $\mathcal{M}$.

Our goal is to prove:

Theorem 5.2. *The theory T^d is complete.*

5.1. **The tight case.** Assume that T is tight. As we saw in Proposition 4.2, the structure $\mathcal{M}^*$ is interpretable in $\mathcal{M}$ without parameters. Using axiom (4) above we immediately conclude:

Lemma 5.3. *Assume that T is tight.*

(1) *If $(\mathcal{M}', \mathcal{M}) \models T^d$ then necessarily $\mathcal{M}' = \mathcal{M}^*$.*
(2) *For all $\mathcal{M}, \mathcal{N} \models T$, we have $(\mathcal{N}^*, \mathcal{N}) \equiv (\mathcal{M}^*, \mathcal{M})$.*

5.2. **The general case.**

Theorem 5.4. *If $\mathcal{M}^d = (\mathcal{M}', \mathcal{M})$ and $\mathcal{N}^d = (\mathcal{N}', \mathcal{N})$ are models of T^d, then $\mathcal{M}^d \equiv \mathcal{N}^d$.*

Proof. We may assume that T is non-tight. We may assume that $\mathcal{M}^d$ and $\mathcal{N}^d$ are κ-saturated for sufficiently large κ.

Notice that every $\mathcal{M}$-definable cut is realized in $\mathcal{M}'$ exactly once, hence there is a natural embedding of $\mathcal{M}^*$ into $\mathcal{M}'$, and the same holds for $\mathcal{N}'$ and $\mathcal{N}$. However, by saturation, unless $\mathcal{M}$ is tight it is not the case that $\mathcal{M}'$ **equals** $\mathcal{M}^*$, since it realizes cuts which are not definable as well. Our goal is to show that there are $(B, A) \prec (\mathcal{M}', \mathcal{M})$ and $(D, C) \prec (\mathcal{N}', \mathcal{N})$ which are isomorphic.

Notice first that both M and $M' \setminus M$ are dense in M', for $i = 1, 2$. Indeed, this follows from the fact that T is non-valuational, so if $c \in \bar{M} \setminus M$ is any element then $c + M \subseteq \bar{M}$ is dense in $\bar{M}$, so also in M'.

Since $\mathcal{M}^* \models T^*$ and $\mathcal{M}^*$ eliminates quantifiers, the pair $(\mathcal{M}', \mathcal{M}^*)$ is an elementary dense pair of o-minimal structures, so we shall apply to it the theory of dense pairs as in [9].

We first need:

Lemma 5.5. *Let $(\mathcal{M}', \mathcal{M}) \models T^d$. Let $M_0 \prec \mathcal{M}$. Then $\mathrm{dcl}_{\mathcal{M}_0^*}(M_0) = \bar{\mathcal{M}}_0$. Moreover, $\mathrm{dcl}_{\mathcal{M}'}(M_0) = \bar{M}_0$.*

Proof. It will suffice to prove the first part of the lemma as the second part follows from the fact that $\mathcal{M}_0^* \prec \mathcal{M}'$.

First we show the right-to-left inclusion. For that we need:

Claim 5.6. *Assume that $f : M_0^n \to \bar{M}_0$ is a $\emptyset$-definable function in $\mathcal{M}_0$. Then there are in $\mathcal{M}_0$ finitely many $\emptyset$-definable strong cells of the form $C_1, \ldots, C_k \subseteq M_0^n$, with $M_0^n \subseteq \bigcup_i \bar{C}_i$, and in $\mathcal{M}_0^*$ there are finitely many $\emptyset$-definable functions $\bar{f}_i : \bar{C}_i \to \bar{M}_0$, such that for all $x \in C_i$, $\bar{f}_i(x) = f(x)$.*

Proof. We decompose M_0^n into $\emptyset$-definable strong cells, $C_1, \ldots, C_k$, on each of which f is strongly continuous. For each i, the graph of $\bar{f} \upharpoonright \bar{C}_i$ is the iterative convex hull of $\Gamma(f \upharpoonright C_i)$, so it is $\emptyset$-definable in $\mathcal{M}_0^*$. □

Assume now that $b \in \bar{M}_0$, then by definition of the completion, the cut $Y = \{x \in M_0 : x < b\}$ is definable in $\mathcal{M}_0$, over a tuple of parameters a. We may assume that $Y = Y_a$ for a $\emptyset$-definable family of sets $\{Y_t : t \in T\}$ and $\emptyset$-definable set $T \subseteq M_0^m$, and that we have $b = \sup Y_a$. It follows that there is in $\mathcal{M}_0$ a $\emptyset$-definable function $f : T \to \bar{M}_0$, such that $f(a) = b$.

By the above claim, we have $T = \bigcup C_i$ a union of $\emptyset$-definable strong cells in $\mathcal{M}$, and there are $f_i : \bar{C}_i \to \bar{M}_0$ all $\emptyset$-definable in $\mathcal{M}_0^*$, , such that

$$\bigwedge_{i=1}^{k} \forall x \in C_i \ \bar{f}(x) = f(x). \tag{5}$$

In particular, there is $i \in \{1, \ldots, k\}$ such that $a \in C_i$ and $b = \bar{f}_i(a)$ is in $\mathrm{dcl}_{\mathcal{M}_0^*}(M_0)$. Thus, $\bar{M}_0 \subseteq \mathrm{dcl}_{\mathcal{M}_0^*}(M_0)$.

For the converse, we assume that $g(a) = b$ for some $\emptyset$-definable function g in $\mathcal{M}_0^*$ and $a \in M_0^m$. We want to show that $b \in \bar{M}_0$, namely that b is the supremum of a definable cut in the structure $\mathcal{M}_0$.

The function g is $\emptyset$-definable in the o-minimal structure $\mathcal{M}_0^*$, so by Theorem 3.3, the set

$$Y = \{(x, y) \in M_0^{n+1} : y < g(x)\}$$

is $\emptyset$-definable in $\mathcal{M}_0$ and we have

$$\forall x \in M_0^n \ g(x) = \sup(Y_x). \tag{6}$$

It follows that $b = g(a) = \sup Y_a$, with $Y \subseteq M_0^{n+1}$ a $\emptyset$-definable set in $\mathcal{M}_0$. Hence, $b \in \bar{M}_0$. □

We will also need:

Claim 5.7. *For $A \subseteq M$ and $a \in M$, the $\mathcal{M}$-type of a over A is determined by the cut of a in $\mathrm{dcl}_{\mathcal{M}'}(A)$.*

Proof. Assume that a and b in M realize the same cut over $\mathrm{dcl}_{\mathcal{M}'}(A)$. To see that a and b realize the same $\mathcal{M}$-type over A, it is sufficient, by the weak o-minimality of $\mathcal{M}$, to show, for every cut $C \subseteq M$ definable in

$\mathcal{M}$ over A, that $a \in C$ iff $b \in C$. Using our assumptions, it is enough to prove that the supremum of C exists in M' and belongs to $\mathrm{dcl}_{\mathcal{M}'}(A)$.

If C has a supremum s in M then $s \in \mathrm{dcl}_{\mathcal{M}}(A) \cap M$, and therefore (Lemma 3.6) $s \in \mathrm{dcl}_{\mathcal{M}^*}(A)$. Since $\mathcal{M}^*$ is an elementary substructure of $\mathcal{M}'$ we have $s \in \mathrm{dcl}_{\mathcal{M}'}(A)$.

If C has no supremum in M then, by definition, its supremum is realized in $\bar{M}$. As C is definable in $\mathcal{M}$ over A, its closure in $\bar{M}$ is $\emptyset$-definable in $\mathcal{M}^*_A$, so by 4.5(2) it is definable in $\mathcal{M}^*_\emptyset$ over A. But then $\sup C \in \mathrm{dcl}_{\mathcal{M}^*}(A) = \mathrm{dcl}_{\mathcal{M}'}(A)$. This finishes the proof. □

The rest of the proof follows closely the arguments from [9]. In order to proceed we borrow the following terminology:

Definition 5.8. For $B \subseteq M'$ and $A = B \cap M$, we say that (B, A) is *free* if $\dim_{\mathcal{M}'}(B'/A) = \dim_{\mathcal{M}'}(B'/M)$ for every finite $B' \subseteq B$. Namely, every subset of B which is $\mathcal{M}'$-independent over A remains independent over M. We make the same definitions for subsets of N' and N.

We consider all $(B, A) \subseteq (M', M)$ (and similarly (D, C) in (N', N)) which satisfy:
(i) $B \cap M = A$.
(ii) $\mathrm{dcl}_{\mathcal{M}'}(B) = B$.
(iii) (B, A) is free.

We now begin the construction of the intended isomorphism. By saturation, there is $\mathcal{M}_0 \prec \mathcal{M}$, of cardinality smaller than κ that is isomorphic to some $\mathcal{N}_0 \prec \mathcal{N}$.

If we let $A_0 := M_0$ $B_0 := \bar{A}_0$ and $C_0 =:= \bar{N}_0$, $D_0 := \bar{C}_0$. Then (i) holds. By Lemma 5.5 $\dim_{\mathcal{M}'}(B_0/A_0) = 0$, so (B_0, A_0) is (trivially) free. Also, by this lemma, B_0 is definably closed in $\mathcal{M}'$, so (B_0, A_0) satisfy (i),(ii),(iii). Similarly, (D_0, C_0) satisfies (i),(ii),(iii).

Our goal is to use back-and-forth and Tarski-Vaught in order to build isomorphic elementary substructures of $(\mathcal{M}', \mathcal{M})$ and $(\mathcal{N}', \mathcal{N})$. Towards that goal we need to prove the following result:

Lemma 5.9. *Assume that $(B, A) \subseteq (\mathcal{M}', \mathcal{M})$ and $(D, C) \subseteq (\mathcal{N}', \mathcal{N})$ satisfy (i),(ii),(iii), and isomorphic (namely, there is an $\mathcal{L}$-isomorphism $\alpha : B \to D$ sending A onto C), with $|A| < \kappa$. Then, for every $b \in M'$, there are $B' \subseteq M', A' \subseteq M$ with $b \in B'$, and there are $D' \subseteq N', C' \subseteq N$, such that (B', A') , (D', C') satisfy (i),(ii),(iii), and there is an isomorphism $\alpha' : (B', A') \to (D', C')$ extending α.*

(We also have the analogous result for (D, C) and $d \in N'$.)

Proof. We divide the argument into several cases:

Case I. $b \in M$.

First, we find $d \in N$ such that $\alpha(\mathrm{tp}_{\mathcal{M}'}(b/B)) = \mathrm{tp}_{\mathcal{N}'}(d/D)$ (so by Lemma 5.7, also $\alpha(\mathrm{tp}_{\mathcal{M}}(b/A)) = \mathrm{tp}_{\mathcal{N}}(d/C)$). Indeed, this is possible because N is dense in N' and $\mathcal{N}'$ is κ-saturated. The function α then extends naturally to an isomorphism α' of the o-minimal structures $B' := \mathrm{dcl}_{\mathcal{M}'}(Bb)$ and $D' := \mathrm{dcl}_{\mathcal{N}'}(Dd)$. We let $A' = B' \cap M$ and $C' = D' \cap N$. In order to see that α' is an isomorphism of (B', A') and (D', C') it is left to verify is that for every $a \in B'$,

$$a \in M \Leftrightarrow \alpha'(a) \in N. \tag{7}$$

So, we take $a \in \mathrm{dcl}_{\mathcal{M}'}(Bb)$ and prove (7).

Assume first that $a \in \mathrm{dcl}_{\mathcal{M}'}(Ab)$. By Lemma 5.5, $a \in \bar{M}$, so we have $a \in \mathrm{dcl}_{\mathcal{M}^*}(Bb)$. Hence, there exists a $\emptyset$-definable function F of $(n+1)$-variables in $\mathcal{M}^*$, and $e \in (\bar{M})^n$, with $F(b, e) = a$. The function F is definable in $\mathcal{M}^*$, and, by 3.3, its restriction to M^{n+1} is $\emptyset$-definable in $\mathcal{M}$ (as a function into $\bar{M}$). Thus, we can definably in $\mathcal{M}$ partition its domain into $\emptyset$-definable strong cells on each of which F takes either values in M or in $\bar{M} \setminus M$. This partition is part of the weakly o-minimal theory T, and thus holds in both $\mathcal{M}$ and $\mathcal{N}$. Since $\alpha(\mathrm{tp}_{\mathcal{M}}(b/A)) = tp_{\mathcal{N}}(d/C)$ it follows that $a = F(b, e) \in M$ if and only if $\alpha'(a) = F(d, \alpha(e)) \in N$.

Assume now that $a \in \mathrm{dcl}_{\mathcal{M}'}(Bb) \setminus \mathrm{dcl}_{\mathcal{M}'}(Ab)$ (so $\alpha'(a) \in \mathrm{dcl}_{\mathcal{N}'}(Dd) \setminus \mathrm{dcl}_{\mathcal{N}'}(Cd)$). We claim that $a \notin M$ and $\alpha'(a) \notin N$.

Indeed, assume towards a contradiction that $a \in M$, and let $Y \subseteq B$ be a minimal finite set which is $\mathrm{dcl}_{\mathcal{M}'}$-independent over Ab such that $a \in \mathrm{dcl}_{\mathcal{M}'}(Y Ab)$. Because $a \notin \mathrm{dcl}_{\mathcal{M}'}(Ab)$ the set Y is nonempty so fix $y_0 \in Y$. We have $a \in \mathrm{dcl}_{\mathcal{M}'}(Y' y_0 Ab)$, with $Y' = Y \setminus \{y_0\}$, so by exchange (and minimality of Y'), $y_0 \in \mathrm{dcl}_{\mathcal{M}'}(Y' Aba)$. Because $a, b \in M$ and $A \subseteq M$, it follows that Y is not independent over M, even though it is independent over A. This contradicts the fact that (B, A) was free, so $a \notin M$. The same argument shows that $\alpha'(a) \notin N$.

Thus, we showed that $\alpha' : (B', A') \to (D', C')$ is an isomorphism. It is clear, that the pairs satisfy (i) and (ii), so we are left to see that they are free. So, we take $Y \subseteq B'$ independent over A' and claim that it remains independent over M. Indeed, because $b \in A'$ (since $b \in M$), it must be the case that $Y \subseteq B$, and the result follows immediately from the freeness of (B, A) (because $A \subseteq A'$). This ends Case I.

Case II. $b \in \mathrm{dcl}_{\mathcal{M}'}(BM)$.

In this case, there is $\bar{m} = (m_1, \ldots, m_k) \in M^k$ such that $b \in \mathrm{dcl}_{\mathcal{M}'}(B\bar{m})$. We first apply Case I to each m_i, and thus may assume that $\bar{m} \subseteq B$, and in particular may assume that b is already in B.

Case III. $b \notin \mathrm{dcl}_{\mathcal{M}'}(BM)$

Notice first that in this case $\mathcal{M}$ (and hence also $\mathcal{N}$) is not tight (since in the tight case $M' = \bar{M} = \mathrm{dcl}_{\mathcal{M}'}(M)$). We let $B' = \mathrm{dcl}_{\mathcal{M}'}(Bb)$ and $A' = B' \cap M$. Our goal is to show that (B', A') satisfies (i),(ii),(iii), so we need to show that it is free.

We first claim that $A' = A$. Indeed, if $a \in \mathrm{dcl}_{\mathcal{M}'}(Bb) \cap M$ then either $a \in \mathrm{dcl}_{\mathcal{M}'}(B)$, so $a \in A$, or if not then by exchange, $b \in \mathrm{dcl}_{\mathcal{M}'}(Ba)$, contradicting the assumption on b.

Assume now that $Y \subseteq B'$ is independent over $A' = A$. If $Y \subseteq B$ then Y is independent over M, and otherwise, we may assume that it is of the form $Y'b$ with $Y' \subseteq B$. By freeness of (B, A) we have Y' independent over M and by assumption on b we may conclude that $Y'b$ independent over M. Thus, (B', A') is indeed free.

Next, we claim that we may find in $\mathcal{N}'$ an element d such that $\alpha(\mathrm{tp}_{\mathcal{M}'}(b/B)) = \mathrm{tp}_{\mathcal{N}'}(d/D)$ and in addition $d \notin \mathrm{dcl}_{\mathcal{N}'}(DN)$. It is here that we use the fact that $\mathcal{N}$ is non-tight. We prove:

Lemma 5.10. *Let $D \subseteq N'$ be of cardinality smaller than κ. Then for every $\mathcal{M}'$-type $p(x)$ over B, there is a realization of $\alpha(p)$ which is not in $\mathrm{dcl}_{\mathcal{N}'}(DN)$.*

Proof. By the saturation of (N', N) it is sufficient to prove that $X \nsubseteq \mathrm{dcl}_{\mathcal{N}'}(DN)$ for every infinite set $X \subseteq N'$ that is definable in $\mathcal{N}'$ over D. For that it is clearly sufficient to show that $X \nsubseteq \mathrm{dcl}_{\mathcal{N}'}(D\bar{N})$. By applying the theory of dense pairs to the pair of o-minimal structures $(\mathcal{N}', \bar{\mathcal{N}})$, we may conclude from [9, Lemma 4.1], that no interval in $\mathcal{N}'$ is in the image of $\bar{N}^n$ under an $\mathcal{N}'$-definable map. This is easily seen to imply the result we want. □

This ends the proof of Lemma 5.9. □

Going back to our proof of completeness of T^d, we find $d \in N'$ with $\alpha(\mathrm{tp}_{\mathcal{M}'}(b/B)) = \mathrm{tp}_{\mathcal{N}'}(d/D)$ and with $d \notin \mathrm{dcl}_{\mathcal{N}'} DN$. We let $D' = \mathrm{dcl}_{\mathcal{N}'}(Dd)$ and $C' = D' \cap N'$ (which equals C), so as before (D', C') is free. It is left to see that the natural extension of α to $\alpha' : B' \to D'$ preserves $M \cap B'$. However, $B' \cap M = A'$ so by applying what we already know to both α and α^{-1} we conclude that $x \in M' \Leftrightarrow \alpha'(x) \in N'$. This ends the proof of Theorem 5.4. □

Notice that the proof above showed that any isomorphism of weakly o-minimal structures $M_1 \prec M$ and $M_2 \prec N$ can be extended to an isomorphism of elementary substructures $(B, A) \prec (M', M)$ and $(D, C) \prec (N', N)$. Lemma 5.9 also implies:

Lemma 5.11. *Assume that $(\mathcal{M}', \mathcal{M}), (\mathcal{N}', \mathcal{N}) \models T^d$ and $(B, A) \subseteq (\mathcal{M}', \mathcal{M})$, $(D, C) \subseteq (\mathcal{N}', \mathcal{N})$ satisfy (i),(ii),(iii). If $\alpha : B \to D$ is an $\mathcal{L}$-isomorphism sending A to C and $\alpha(b) = d$ for some $b \in B^n$ then*

$$\alpha(\mathrm{tp}_{(\mathcal{M}', \mathcal{M})}(b/\emptyset)) = \mathrm{tp}_{(\mathcal{N}', \mathcal{N})}(d/\emptyset).$$

We can now prove analogues of several theorems from [9]. The proofs are very similar to the original ones.

Theorem 5.12. *Let $\mathcal{M}^d = (\mathcal{M}', \mathcal{M})$ be a model of T^d.*

(1) *In $\mathcal{M}^d$, every $\emptyset$-definable subset of $(M')^n$ is a boolean combination of sets defined by formulas of the form*

$$\exists x_1 \cdots \exists x_k (\bigwedge_{i=1}^{k} x_i \in P \,\&\, \varphi(x_1, \ldots, x_k, y), \tag{8}$$

where $|y| = n$ and $\varphi(x, y)$ is an $\mathcal{L}$ formula.

(2) *Let $B \subseteq M'$ be such that $(B, B \cap M)$ is free. Then every subset of M^k that is definable in $\mathcal{M}^d$ over $B \subseteq M'$ is of the form $Y \cap M^k$ for some $Y \subseteq (M')^k$ that is definable in $\mathcal{M}'$ over B.*

(3) *Every subset of M^k that is definable in $\mathcal{M}^d$ over $A_0 \subseteq M$ is definable in the structure $\mathcal{M}$ over A_0.*

(4) *Every subset of M^n that is definable in $(\mathcal{M}^*, \mathcal{M})$ (here $\mathcal{M}^*$* **is** *the completion of $\mathcal{M}$) is definable in the structure $\mathcal{M}$.*

Proof. Without loss of generality, $(\mathcal{M}', \mathcal{M})$ is sufficiently saturated.

(1) By standard model theoretic considerations it is enough to prove the following: For any $b, d \in (M')^k$, assume that b satisfies a formula of the form (8) if and only if d does. Then b and d have the same type in $\mathcal{M}^*$ over $\emptyset$.

Let $r = \dim_{\mathcal{M}'}(b/M)$. We can find $a \subseteq M$ finite such that $\dim_{\mathcal{M}'}(b/a) = r$. It follows that if we let $B = \mathrm{dcl}_{\mathcal{M}'}(ab)$ and $A = \mathrm{dcl}_{\mathcal{M}'}(a)$ then (B, A) is free and $A = B \cap M$.

We consider the $\mathcal{L}$-type of (b, a) over $\emptyset$. Because b and d realize the same formulas of the form (8), and because of saturation we can find $c \in M$ such that $\mathrm{tp}_{\mathcal{M}'}(b, a/\emptyset) = \mathrm{tp}_{\mathcal{M}'}(d, c/\emptyset)$. The pair (D, C), with $D = \mathrm{dcl}_{\mathcal{M}'}(cd)$ and $C = \mathrm{dcl}_{\mathcal{M}'}(c)$ is free with $C = D \cap M$. Just like in the proof of Lemma 5.9, the natural $\mathcal{L}$-isomorphism of B and D (sending (b, a) to (d, c)) sends A to C.

By Lemma 5.11, the $\mathcal{L}^P$-types of b and d in $(\mathcal{M}', \mathcal{M})$ are the same. Thus we proved (1).

(2) By standard model theoretic arguments it is sufficient to prove: If $b_1, b_2 \in M^k$ satisfy the same $\mathcal{M}'$-type over B then they satisfy the same L^P-type over B. For that, let $A = B \cap M$. It is sufficient to show that there are $(B_1, A_1), (B_2, A_2) \prec (\mathcal{M}', \mathcal{M})$, with $(B, A) \subseteq (B_i, A_i)$ and $b_i \in B_i$ for $i = 1, 2$, and there is an $\mathcal{L}$-isomorphism between (B_1, A_1) and (B_2, A_2), which fixes B point-wise, and sending b_1 to b_2.

We are now in the setting of Case I of the proof of Lemma 5.9, with our b_1, b_2 replacing b, d there. Thus, we may first find two free pairs (B_1', A_1') and (B_2', A_2') with $B \subseteq B_i'$ and $b_i \in B_i'$, $i = 1, 2$, and an isomorphism $\alpha : (B_1', A_1') \to (B_2', A_2')$ extending the identity map, with $\alpha(b_1) = b_2$. We now proceed exactly as in the proof of Theorem 5.4 and obtain the desired $(B_1, A_1), (B_2, A_2) \prec (\mathcal{M}', \mathcal{M})$. Thus, b_1 and b_2 realize the same $\mathcal{L}^P$ type over B and we may conclude (1).

For (3), let $X \subseteq M^k$ be definable in $(\mathcal{M}', \mathcal{M})$ over $A_0 \subseteq M$. Notice that the mere definability of X in $\mathcal{M}$ follows immediately from (2) but we want to show that X is definable over the same A_0. For that, it is sufficient to prove that any $a_1, a_2 \in M$ which realize the same $\mathcal{M}$-type over A_0 realize the same $\mathcal{L}^P$-type over A_0.

To do that, we first find a small model $\mathcal{M}_1 \prec \mathcal{M}$ containing $A_0\, a_1, a_2$, and an automorphism α of $\mathcal{M}_1$ over A_0, sending a_1 to a_2. As we commented previously, we may the extend α to an isomorphism of two structures $(B, A), (D, C) \prec (\mathcal{M}', \mathcal{M})$. This is clearly sufficient.

To see (4), we note that every element of $\mathcal{M}^*$ is in $\mathrm{dcl}_{\mathcal{M}^*}(N)$ and hence every definable subset of $\bar{M}^k$ in $\mathcal{M}^*$ can be defined over M. We now apply (3). □

Note that (3) above fails if we omit the requirement that $M_0 \subseteq M$, since in the non-tight case, in general, $\mathcal{M}'$ will realize cuts which are not definable in $\mathcal{M}$ and thus their intersection with M is not definable in $\mathcal{M}$.

We also point out:

Lemma 5.13. *If $\mathcal{M}^d = (\mathcal{M}', \mathcal{M}) \models T^d$ then it is definably complete.*

Proof. If $X \subseteq M'$ is definable in $\mathcal{M}^d$ and bounded below then the intersection of its convex hull with M is definable in $\mathcal{M}^d$, and thus has the form $Y \cap M$ for some $Y \subseteq M'$ which is definable in $\mathcal{M}'$. Without loss of generality, Y is also convex and thus $\mathrm{Inf}Y = \mathrm{Inf}X$. This suffices, by o-minimality of $\mathcal{M}'$. □

We can now conclude, using Boxall and Hieronymi, [2]:

Theorem 5.14. *Let $\mathcal{M}^d = (\mathcal{M}', \mathcal{M}) \models T^d$. If $U \subseteq (M')^n$ is open and definable in $\mathcal{M}^d$ then it is definable in $\mathcal{M}'$. More precisely, if an open U is defined in $\mathcal{M}^d$ over $B \subseteq M'$ such that $(B, B \cap M)$ is free, then U is definable in $\mathcal{M}'$ over B. In particular, $\mathcal{M}^P$ has an o-minimal open core.*

Proof. This is an immediate corollary of [2, Corollary 3.2] and what we proved so far. We extract from their argument a direct proof, which is underlined by the following simple corollary of cell decomposition.

Fact 5.15. *If $Y \subseteq (M')^n$ is definable in $\mathcal{M}'$ and $\dim Y < n$ then $Y \cap M^n$ has empty interior in M^n.*

We now first claim that $\mathrm{cl}_{M'}(U)$ is definable in $\mathcal{M}'$ over B. Indeed, by Theorem 5.12 (2), there is $Y \subseteq (M')^n$ definable in $\mathcal{M}'$ over B such that $Y \cap M^n = U \cap M^n$. By the above observation, $\dim Y = n$.

Since M^n is dense in $(M')^n$, the set $Int(Y) \cap M^n$ is dense in the open set $Int(Y)$. We claim that it is also dense in U. Indeed, we know that $Y \cap M^n = U \cap M^n$ is open in M^n and dense in U, and by o-minimality $\dim_{\mathcal{M}'}(Y \setminus Int(Y)) < n$. It thus follows from Fact 5.15, that $Int(Y) \cap M^n$ is dense in U.

So,

$$\mathrm{cl}_{M'}(U) = \mathrm{cl}_{M'}(Int(Y) \cap M^n) = \mathrm{cl}_{M'}(Int(Y)).$$

Because Y was definable in $\mathcal{M}'$ over B, $\mathrm{cl}_{M'}(U)$ is definable in $\mathcal{M}'$ over B.

We thus showed that the closure of every $\mathcal{M}^d$-definable open set over $B \subseteq M'$ is definable in $\mathcal{M}'$ over B. It follows that every $\mathcal{M}^d$-definable continuous function $f : (M')^n \to M$ is definable in $\mathcal{M}'$, over the same parameters. Indeed, the closure of the open set $\{(x, y) \in (M')^{n+1} : y < f(x)\}$ is exactly $\{(x, y) \in (M')^{n+1} : y \leq f(x)\}$, from which the definability of f follows.

Finally, we show that every closed $F \subseteq (M')^n$ set which is $\mathcal{M}^d$-definable over $B \subseteq M'$ is definable in $\mathcal{M}'$ over B. For every $x \in M^n$ we let $f(x) = d(x, F) = Inf\{d(x, y) : y \in F\}$. By Lemma 5.13, this is a well defined function in $\mathcal{M}^d$ (over B), and since F is closed, the function f is continuous and F is its zero set. Because f is definable in $\mathcal{M}'$ over B, so is the set F.

Since every definable set in $\mathcal{M}'$ can be defined over some $B \subseteq M'$ with $(B, B \cap M)$ free, the theorem follows. □

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