Locally Moving Polymophism Clones

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Set Theory, Model Theory and Applications
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In memory of Mati Rubin

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Motivation

- Reconstruction from Polymorphism Clones

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- Automatic Homeomorphicity

Automatic Homeomorphicity

Definition

Let M be a first order structure and let $\mathrm{Pol}(M)$ denote its polymorphism clone. Let K be a class of polymorphism clones. $\mathrm{Pol}(M)$ is said to have *automatic homeomorphicity* with respect to K if for all $N \in K$, if there is an isomorphism $\theta : M \to N$ then θ is a homeomorphism with respect to the topology of pointwise convergence.

Motivation

- Reconstruction from Polymorphism Clones
- Automatic Homeomorphicity
- Constraint Satisfaction Problems

Constraint Satisfaction Problems

If there is a $\theta : \mathsf{Pol}(M) \to \mathsf{Pol}(N)$ such that θ is both an isomorphism and a homeomorphism
then $\mathrm{CSP}(M)$ and $\mathrm{CSP}(N)$ are polynomial time equivalent.

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- Rubin

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Rubin's Locally Moving Theorem

Theorem (Rubin's Locally Moving Theorem)

Let G and H be locally moving groups, witnessed by $B(G)$ and $B(H)$ respectively, and let $\theta : G \to H$ be an isomorphism. Then there is an isomorphism $\tau : B(G) \to B(H)$ such that

$$\theta(g) = \tau g \tau^{-1}$$

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- Being locally moving is a first order property.
- Conjugation and centralisers play a vital role in the proof.

Application to Reconstruction

- Boolean Algebras,
- Many kinds of Orders, including:
  - Reducts of some linear orders
  - Reducts of some semi-linear orders
  - (Conjecture) NIP partial orders with a sufficently large automorphism group
- Many Topological Spaces
- Graph Trees

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Topolgical Spaces	Basic Open Sets

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Let M be a first order structure.

- $\mathrm{Emb}(M)$ is the monoid of self-embeddings.
- $\mathrm{End}(M)$ is the endomorphism monoid.
- $\mathrm{Pol}(M)$ is the polymorphism clone.

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Let M be a first order structure. We say that Pol(M) is *locally moving* if and only if

- Aut(M) is locally moving, witnessed by $B(M)$, and
- For every $f \in \mathrm{End}(M)$ and $\alpha \in$ Aut(M) there are $g \in \mathrm{Emb}(M)$ and $\beta, \gamma \in$ Aut(M) such that:
  - $\mathrm{var}(\alpha) = \mathrm{var}(\gamma)$,
  - $gf\gamma = \beta gf$.

Remarks

Second Condition

For every $f \in \mathrm{End}(M)$ and $\alpha \in \mathsf{Aut}(M)$ there are $g \in \mathrm{Emb}(M)$ and $\beta, \gamma \in \mathsf{Aut}(M)$ such that:

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- Allows an ersatz conjugation.
- Condition is only on the monoid.
- Implies $\mathrm{End}(M) = \mathrm{Emb}(M)$.

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If M and N are such that $\mathrm{Pol}(M)$ is locally moving, and $\theta : \mathrm{Pol}(M) \to \mathrm{Pol}(N)$ is an isomorphism, then θ is a homeomorphism with respect to the topology of pointwise convergence.

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- Extend Rubin's formulas to $\mathrm{Pol}(M)$ via projections.
- Show that the topologies from the actions on M and $B(\mathrm{Aut}(M))$ are the same.

Examples of LM Clones

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- Pol(M) if M is a reduct of $(\mathbb{S}_2, <)$ such that $\mathrm{End}(M) = \mathrm{Emb}(M)$.
- Pol(M) if M is a reduct of the generic Boolean Algebra such that $\mathrm{End}(M) = \mathrm{Emb}(M)$.

$\mathsf{Pol}(\mathbb{Q}, <)$

Theorem

If M is a reduct of $(\mathbb{Q}, <)$ such that $\mathrm{End}(M) = \mathrm{Emb}(M)$ then $\mathsf{Pol}(M)$ *is locally moving.*

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First we will prove that $\mathrm{Aut}(M)$ is locally moving. Let B' be the sub-algebra of $\mathcal{P}(\mathbb{Q})$ generated by the open intervals.

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$$I_1 \sim I_2 :\Leftrightarrow \mathrm{int}(\mathrm{cl}(I_1)) = \mathrm{int}(\mathrm{cl}(I_2))$$

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$B := B' / \sim$

B is complete and atomless. Let $b \in B$. Then there is an open interval $(a_1, a_2) \subseteq b$. Then there is an $\alpha \in \text{Aut}(\mathbb{Q}, <) \leq \text{Aut}(M)$ such that

$$\text{supp}(\alpha) = \mathrm{var}(\alpha) \subseteq (a_1, a_2) \subseteq b$$

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Therefore $\mathsf{Aut}(M)$ is locally moving.

The Second Condition

For every $f \in \mathrm{End}(M)$ and $\alpha \in \mathsf{Aut}(M)$ there are $g \in \mathrm{Emb}(M)$ and $\beta, \gamma \in \mathsf{Aut}(M)$ such that:

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The same g works for all f and all α, and we can always take $\gamma := \alpha$. This is because $\mathbb{Q}$ is very nice.

Let $\mathbb{Q}_2$ be the Fraïssé generic $\{\mathrm{Red}, \mathrm{Blue}\}$-coloured linear order.
Let

$$g' : \mathbb{Q} \to \{q \in \mathbb{Q} \,:\, \mathbb{Q}_2 \models \mathrm{Red}(q)\}$$

be an isomorphism.

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be an isomorphism.
$(\mathbb{Q}_2 \times \mathbb{Q}, <_{\mathrm{lex}}) \cong \mathbb{Q}$
$g : \mathbb{Q} \to (\mathbb{Q}_2 \times \mathbb{Q})$ is given by $g(q) = (g'(q), 0)$.

References

- M. Rubin
  Locally moving groups and reconstruction problems
  Ordered Groups and Infinite Permutation Groups, pages 121-157
  Kluwer Academic Publishing, 1996.
- R. Barham
  Automatic Homeomorphicity of Locally Moving Clones
  arXiv:1512.00251