Uniqueness triples from the diamond axiom

Ari Meir Brodsky ©

Ariel University

11 Iyyar, 5778
Thursday, April 26, 2018
Set Theory, Model Theory and Applications
Ben-Gurion University of the Negev, Eilat Campus

This is joint work with Adi Jarden.

This research was carried out with the assistance of the Center for Absorption in Science, Ministry of Aliyah and Integration, State of Israel.

Preliminaries

Fix a vocabulary τ.

Preliminaries

Fix a vocabulary τ.
We will consider a class of models $\mathbf{K}$ for τ,
along with a partial-order relation $\preceq_{\mathbf{K}}$ on $\mathbf{K}$.

Preliminaries

Fix a vocabulary τ.
We will consider a class of models $\mathbf{K}$ for τ,
along with a partial-order relation $\preceq_{\mathbf{K}}$ on $\mathbf{K}$.
$\preceq_{\mathbf{K}}$ will be a refinement of the submodel relation $\subseteq$.

Galois types

For a model $A \in \mathbf{K}$ and a point $b \notin A$, we want to be able to describe the type of b over A, that is, the relationship between b and A.

Galois types

For a model $A \in \mathbf{K}$ and a point $b \notin A$, we want to be able to describe the type of b over A, that is, the relationship between b and A.
To do so, we fix an ambient model $B \in \mathbf{K}$ that contains b.
That is, for $A, B \in \mathbf{K}$ with $A \preceq_{\mathbf{K}} B$ and $b \in B \setminus A$, we write

$$\mathrm{tp}(b/A, B)$$

for the Galois-type of b over A, where b is viewed as an element of B.

Galois types

For a model $A \in \mathbf{K}$ and a point $b \notin A$, we want to be able to describe the type of b over A, that is, the relationship between b and A.
To do so, we fix an ambient model $B \in \mathbf{K}$ that contains b.
That is, for $A, B \in \mathbf{K}$ with $A \preceq_{\mathbf{K}} B$ and $b \in B \setminus A$, we write

$$\mathrm{tp}(b/A, B)$$

for the Galois-type of b over A, where b is viewed as an element of B.
When should two triples be considered to have the same type?

Galois types

Definition

We say that (A, b, B) and (A, c, C) have the same type if there exist model D and embeddings $f : B \hookrightarrow D$ and $g : C \hookrightarrow D$ that are constant on A and such that $f(b) = g(c)$.

(This is an equivalence relation provided the Amalgamation Property holds.)

Galois types

Definition

We say that (A, b, B) and (A, c, C) have the same type if there exist model D and embeddings $f : B \hookrightarrow D$ and $g : C \hookrightarrow D$ that are constant on A and such that $f(b) = g(c)$.

(This is an equivalence relation provided the Amalgamation Property holds.)
If there is a monster model M in which all models lie, then: (A, b, B) and (A, c, C) have the same type iff there is an automorphism of M that fixes A and sends b to c.

What classes of models do we consider?

Definition

And abstract elementary class is a class **K** of models for τ, with relation $\preceq_{\mathbf{K}}$, satisfying:

- **K** and $\preceq_{\mathbf{K}}$ respect isomorphisms;
- $\preceq_{\mathbf{K}}$ is a partial order that refines $\subseteq$;
- If $\langle M_\alpha \mid \alpha < \delta \rangle$ is a $\preceq_{\mathbf{K}}$-increasing sequence, then
$$M_0 \preceq_{\mathbf{K}} \bigcup \{M_\alpha \mid \alpha < \delta\} \in \mathbf{K};$$
- If $\langle M_\alpha \mid \alpha \leq \delta \rangle$ is a $\preceq_{\mathbf{K}}$-increasing, continuous sequence and $M_\alpha \preceq_{\mathbf{K}} N$ for all $\alpha < \delta$, then $M_\delta \preceq_{\mathbf{K}} N$;
- If $A \subseteq B \subseteq C$, $A \preceq_{\mathbf{K}} C$, and $B \preceq_{\mathbf{K}} C$, then $A \preceq_{\mathbf{K}} B$;
- There is a Löwenheim-Skolem-Tarski number for **K**: the first infinite cardinal λ such that for every $N \in \mathbf{K}$ and every subset $Z \subseteq N$, there is $M \in \mathbf{K}$ such that $Z \subseteq M \preceq_{\mathbf{K}} N$ and $|M| \leq \lambda + |Z|$.

Examples of Abstract Elementary Classes

- Let T be a first-order theory. Denote $\mathbf{K} := \{M \mid M \models T\}$. Define $M \preceq_{\mathbf{K}} N$ iff M is an elementary submodel of N. Then $(\mathbf{K}, \preceq_{\mathbf{K}})$ is an AEC.

Examples of Abstract Elementary Classes

- Let T be a first-order theory. Denote $\mathbf{K} := \{M \mid M \models T\}$. Define $M \preceq_{\mathbf{K}} N$ iff M is an elementary submodel of N. Then $(\mathbf{K}, \preceq_{\mathbf{K}})$ is an AEC.
- Let T be a first-order theory with π_2 axioms, that is, axioms of the form $\forall x \exists y \varphi(x, y)$, where φ is quantifier-free. Denote $\mathbf{K} := \{M \mid M \models T\}$. Then $(\mathbf{K}, \subseteq)$ is an AEC.

Examples of Abstract Elementary Classes

- Let T be a first-order theory. Denote $\mathbf{K} := \{M \mid M \models T\}$. Define $M \preceq_{\mathbf{K}} N$ iff M is an elementary submodel of N. Then $(\mathbf{K}, \preceq_{\mathbf{K}})$ is an AEC.
- Let T be a first-order theory with π_2 axioms, that is, axioms of the form $\forall x \exists y \varphi(x, y)$, where φ is quantifier-free. Denote $\mathbf{K} := \{M \mid M \models T\}$. Then $(\mathbf{K}, \subseteq)$ is an AEC.
- The class of locally finite groups with the relation $\subseteq$ is an AEC.

Equivalent amalgamations

When are two amalgamations essentially the same?

Equivalent amalgamations

When are two amalgamations essentially the same?
When you can amalgamate them together in such a way that the diagram commutes. . . .

Domination triples

Definition

Suppose $A, B \in \mathbf{K}_\lambda$ with $A \preceq_{\mathbf{K}} B$, and $b \in B \setminus A$. We say that b *dominates* B *over* A and that (A, b, B) is a *domination triple* if for every $C \in \mathbf{K}_\lambda$ such that $A \preceq_{\mathbf{K}} C$, and any two amalgamations $(f_1^D, \mathrm{id}_C, D)$ and $(f_1^E, \mathrm{id}_C, E)$ of B and C over A that are not equivalent over A, if $f_1^D(b), f_1^E(b) \notin C$, then $\mathrm{tp}(f_1^D(b)/C, D) \neq \mathrm{tp}(f_1^E(b)/C, E)$.

Domination triples

Definition

Suppose $A, B \in \mathbf{K}_\lambda$ with $A \preceq_{\mathbf{K}} B$, and $b \in B \setminus A$. We say that b *dominates* B *over* A and that (A, b, B) is a *domination triple* if for every $C \in \mathbf{K}_\lambda$ such that $A \preceq_{\mathbf{K}} C$, and any two amalgamations $(f_1^D, \mathrm{id}_C, D)$ and $(f_1^E, \mathrm{id}_C, E)$ of B and C over A that are not equivalent over A, if $f_1^D(b), f_1^E(b) \notin C$, then $\mathrm{tp}(f_1^D(b)/C, D) \neq \mathrm{tp}(f_1^E(b)/C, E)$.

Example

$\mathbf{K}$ = class of fields
$\preceq_{\mathbf{K}}$ = subfield
Then (A, b, B) is a dominating triple if $B = \mathrm{cl}(A \cup \{b\})$.

Adding a non-forking relation

We want to add a *non-forking relation* $\perp$ to our structures:

Adding a non-forking relation

We want to add a *non-forking relation* $\downarrow$ to our structures: $\downarrow(A, B, c, C)$, where $A, B, C \in \mathbf{K}$ with $A \preceq_{\mathbf{K}} B \preceq_{\mathbf{K}} C$ and $c \in C \setminus B$.

Adding a non-forking relation

We want to add a *non-forking relation* $\mathop{\smile\!\!\!\!\!\!\;|}$ to our structures: $\mathop{\smile\!\!\!\!\!\!\;|}(A, B, c, C)$, where $A, B, C \in \mathbf{K}$ with $A \preceq_{\mathbf{K}} B \preceq_{\mathbf{K}} C$ and $c \in C \setminus B$.
The relation needs be invariant under isomorphisms, and also under type equivalence, and satisfy certain monotonicity requirements.

Adding a non-forking relation

We want to add a *non-forking relation* $\downfree$ to our structures: $\downfree(A, B, c, C)$, where $A, B, C \in \mathbf{K}$ with $A \preceq_{\mathbf{K}} B \preceq_{\mathbf{K}} C$ and $c \in C \setminus B$.

The relation needs be invariant under isomorphisms, and also under type equivalence, and satisfy certain monotonicity requirements.

If $\downfree(A, B, c, C)$, we say that $\mathrm{tp}(c/B, C)$ does not fork over A.

Adding a non-forking relation

We want to add a *non-forking relation* $\downarrow$ to our structures: $\downarrow(A, B, c, C)$, where $A, B, C \in \mathbf{K}$ with $A \preceq_{\mathbf{K}} B \preceq_{\mathbf{K}} C$ and $c \in C \setminus B$.

The relation needs be invariant under isomorphisms, and also under type equivalence, and satisfy certain monotonicity requirements.

If $\downarrow(A, B, c, C)$, we say that $\mathrm{tp}(c/B, C)$ does not fork over A.

If $\downarrow(A, A, c, C)$, we say that $\mathrm{tp}(c/A, C)$ is a basic type, denoted $\mathrm{tp}(c/A, C) \in S^{\mathrm{bs}}(A)$.

Uniqueness triples

Definition

Suppose $A, B \in \mathbf{K}_\lambda$ with $A \preceq_{\mathbf{K}} B$, and $b \in B \setminus A$. We say that (A, b, B) is a *uniqueness triple* if $\mathrm{tp}(b/A, B) \in S^{\mathrm{bs}}(A)$ and for every $C \in \mathbf{K}_\lambda$ such that $A \preceq_{\mathbf{K}} C$, and any two amalgamations $(f_1^D, \mathrm{id}_C, D)$ and $(f_1^E, \mathrm{id}_C, E)$ of B and C over A that are not equivalent over A, it cannot be that both $\mathrm{tp}(f_1^D(b)/C, D)$ and $\mathrm{tp}(f_1^E(b)/C, E)$ do not fork over A.

Uniqueness triples

Definition

Suppose $A, B \in \mathbf{K}_\lambda$ with $A \preceq_{\mathbf{K}} B$, and $b \in B \setminus A$. We say that (A, b, B) is a *uniqueness triple* if $\mathrm{tp}(b/A, B) \in S^{\mathrm{bs}}(A)$ and for every $C \in \mathbf{K}_\lambda$ such that $A \preceq_{\mathbf{K}} C$, and any two amalgamations $(f_1^D, \mathrm{id}_C, D)$ and $(f_1^E, \mathrm{id}_C, E)$ of B and C over A that are not equivalent over A, it cannot be that both $\mathrm{tp}(f_1^D(b)/C, D)$ and $\mathrm{tp}(f_1^E(b)/C, E)$ do not fork over A.

Example

Equivalence classes. . . .

Why are uniqueness triples important?

Uniqueness triples allow us to move from a good λ-frame to a good λ^+-frame, and thus to deduce existence of models of cardinality λ^{+++}.

Prior results

Theorem (Shelah)

Suppose that:

1. $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$;
2. $\mathfrak{s}$ *is a good* λ*-frame;*
3. $\dot{I}(\lambda^{++}, \mathbf{K}) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^\lambda) \sim 2^{\lambda^{++}}$

Then every basic triple can be extended to a uniqueness triple.

New result

Theorem (Main Theorem, B. & Jarden, 2018)

Suppose that:

1. λ *is an infinite cardinal such that* $\diamondsuit(\lambda^+)$ *holds;*
2. $\mathfrak{s} = (\mathbf{K}, \preceq_{\mathbf{K}}, S^{\mathrm{bs}}, \bigcup)$ *is a good* λ*-frame;*
3. $A \in \mathbf{K}_\lambda$*;*
4. $p \in S^{\mathrm{bs}}(A)$.

Then there exist models $C, D \in \mathbf{K}_\lambda$ *such that* $A \prec_{\mathbf{K}} C \prec_{\mathbf{K}} D$ *and* $b \in D \setminus C$ *such that:*

1. (C, D, b) *is a uniqueness triple;*
2. $\mathrm{tp}(b/A, D) = p$*; and*
3. $\bigcup(A, C, b, D)$.

A more useful form of $\diamondsuit$

Jensen (1972) introduced the $\diamondsuit$ axiom to predict subsets of κ.
But usually what we want to guess are subsets of some structure of size κ, not necessarily sets of ordinals.
Encoding the desired sets as sets of ordinals is cumbersome, and distracts us from properly applying the guessing power of $\diamondsuit$.

A more useful form of $\diamondsuit$

Jensen (1972) introduced the $\diamondsuit$ axiom to predict subsets of κ.
But usually what we want to guess are subsets of some structure of size κ, not necessarily sets of ordinals.
Encoding the desired sets as sets of ordinals is cumbersome, and distracts us from properly applying the guessing power of $\diamondsuit$.

Definition (B. & Rinot, 2017)

$\diamondsuit^-(H_\kappa)$ asserts the existence of a sequence $\langle \Omega_\beta \mid \beta < \kappa \rangle$ of elements of H_κ such that for every parameter $z \in H_{\kappa^+}$ and every subset $\Omega \subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M} \prec_{\mathsf{FO}} H_{\kappa^+}$ with $z \in \mathcal{M}$, such that $\kappa^{\mathcal{M}} := \mathcal{M} \cap \kappa$ is an ordinal $< \kappa$ and $\mathcal{M} \cap \Omega = \Omega_{\kappa^{\mathcal{M}}}$.

Here, H_θ denotes the collection of all sets of hereditary cardinality less than θ.

A more useful form of $\diamondsuit$

Jensen (1972) introduced the $\diamondsuit$ axiom to predict subsets of κ.
But usually what we want to guess are subsets of some structure of size κ, not necessarily sets of ordinals.
Encoding the desired sets as sets of ordinals is cumbersome, and distracts us from properly applying the guessing power of $\diamondsuit$.

Definition (B. & Rinot, 2017)

$\diamondsuit^-(H_\kappa)$ asserts the existence of a sequence $\langle \Omega_\beta \mid \beta < \kappa \rangle$ of elements of H_κ such that for every parameter $z \in H_{\kappa^+}$ and every subset $\Omega \subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M} \prec_{\mathsf{FO}} H_{\kappa^+}$ with $z \in \mathcal{M}$, such that $\kappa^{\mathcal{M}} := \mathcal{M} \cap \kappa$ is an ordinal $< \kappa$ and $\mathcal{M} \cap \Omega = \Omega_{\kappa^{\mathcal{M}}}$.

Here, H_θ denotes the collection of all sets of hereditary cardinality less than θ.

Proposition (B. & Rinot, 2017)

For any regular uncountable cardinal κ, $\diamondsuit(\kappa) \iff \diamondsuit^-(H_\kappa)$.

Proof of Main Theorem

Sketch of the proof — on the board...

References

Ari Meir Brodsky and Adi Jarden.
Uniqueness triples from the diamond axiom.
Preprint, arXiv:1804.10952, April 2018.
https://arxiv.org/abs/1804.10952

Ari Meir Brodsky and Assaf Rinot.
A Microscopic approach to Souslin-tree constructions. Part I.
Annals of Pure and Applied Logic, 168(11): 1949–2007, 2017.

Saharon Shelah.
Classification Theory for Abstract Elementary Classes 2.
Studies in Logic: Mathematical logic and foundations, College Publications, 2009.

Saharon Shelah.
Non-structure in λ^{++} using instances of WGCH.
Chapter VII, in series Studies in Logic, volume 20, College Publications. Sh:838. arxiv:0808.3020.