Infinitary methods in finite combinatorics

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Vague idea: we can answer questions about large finite structures using infinitary methods.

Sample questions: How many monochromatic triangles are there in a 3-colouring of the edges of a large complete graph? How close is a graph with few triangles to a graph with no triangles?

This talk is about two (closely related) ideas: Flag algebras (Razborov), structural limits (Lovasz and Szegedy).

L is a finite relational language, T is a universal theory with an infinite model. $\mathcal{M}_n$ is the (nonempty, finite) set of models of size n up to isomorphism. If $M \in \mathcal{N}$ and $A \subseteq |M|$ then $M|A$ is the induced substructure.

Density: M and N are finite models. $p(M, N)$ is the probability that $N|A \simeq M$, where $A \subseteq |N|$ is a set of size $\|M\|$ chosen uniformly at random. By convention $p(M, N) = 0$ when M is bigger than N.

Chain rule: If $r \leqslant s \leqslant t$, $A \in \mathcal{M}_r$ and $C \in \mathcal{M}_t$, then $p(A, C) = \sum_{B \in \mathcal{M}_s} p(A, B)p(B, C)$.

For purposes of measuring density in large structures, "$A = \sum_B p(A, B)B$".

A sequence of finite models (M_n) is *increasing* if $\|M_n\| \to \infty$, and an increasing sequence is *convergent* if $(p(A, M_n))$ converges to a limit for every finite model A.

It is easy to see that any increasing sequence has a convergent subsequence.

Idea 1: assign to each convergent sequence some kind of limit. These limit objects will be algebraic in the theory of flag algebras, analytic in the theory of structural limits.

Idea 2: Infinitary reasoning about the limit objects translates to assertions about large finite models.

Let (M_n) be convergent and define a function Φ on finite models by $\Phi(A) = \lim_{n\to\infty} p(A, M_n)$.

By the Chain Rule, if $A \in \mathcal{M}_r$ and $r \leqslant s$, then $\Phi(A) = \sum_{B\in\mathcal{M}_s} p(A, B)\Phi(B)$.

Let $\mathbb{R}\mathcal{M}$ be the free real VS with basis $\bigcup_n \mathcal{M}_n$ and let $\mathcal{K}$ be the subspace generated by all expressions of the form $A - \sum_{B\in\mathcal{M}_s} p(A, B)B$ for $A \in \mathcal{M}_r$ and $r \leqslant s$. Let $\mathcal{F}$ be the quotient space $\mathbb{R}\mathcal{M}/\mathcal{K}$.

$\mathcal{F}$ is the *flag algebra* for T. If we extend Φ to a linear map from $\mathbb{R}\mathcal{M}$ then Φ vanishes on $\mathcal{K}$, so as usual we get well-defined Φ from $\mathcal{F}$ to $\mathbb{R}$.

So far $\mathcal{K}$ is only a VS. We want to make it into an algebra in such a way that Φ induced by a convergent sequence becomes an algebra HM.

Key point: If X is a large set and we randomly choose two small sets, they are very probably disjoint.

Let $M_1 \in \mathcal{M}_r$ and $M_2 \in \mathcal{M}_s$. For $N \in \mathcal{M}_t$, $p(M_1, M_2; N)$ is the probability that choosing disjoint sets $A_1, A_2 \subseteq |N|$ with $|A_i| = \|M_i\|$ gives $N|A_i \simeq M_i$. Zero if $t < r + s$.

Choose $t \geqslant r + s$, and define $M_1 \cdot M_2 = \sum_{N \in \mathcal{M}_t} p(M_1, M_2; N)N$. One can verify that this induces a well-defined multiplication operation on $\mathcal{F}$ (which is spanned as a VS by the cosets of finite models).

Theorem (Razborov):
A) If (M_n) is convergent and $\Phi : \mathcal{F} \to \mathbb{R}$ is the induced map, then Φ is an $\mathbb{R}$-algebra HM from $\mathcal{K}$ to $\mathbb{R}$.
B) If Φ is an $\mathbb{R}$-algebra HM from $\mathcal{K}$ to $\mathbb{R}$ such that $\Phi(A) \geqslant 0$ for every finite model A, then Φ is induced by some convergent sequence.

A) is a kind of Soundness Theorem, B) is a kind of Completeness Theorem.

Hints for the proof of B). Φ induces a probability distribution on $\mathcal{M}_{n^2}$, choose $\mathbf{M}_n$ accordingly. Want to show that almost surely $(\mathbf{M}_n)$ converges and induces Φ. Estimate the mean and variance of $p(A, \mathbf{M}_n)$, then use the Chebyshev bound and the Borel-Cantelli Lemma to show that almost surely $p(A, \mathbf{M}_n) \to \Phi(A)$.

Let $A \in \mathcal{F}$ and $r \in \mathbb{R}$. Say that $A \geqslant r$ if $\Phi(A) \geqslant r$ for all Φ as above.

Easy: $A \geqslant r$ if and only if for all ϵ for all sufficiently large models N we have $p(A, N) > r - \epsilon$.

Easy: The class of the empty structure is the 1 of $\mathcal{F}$, and for any n we have $1 = \sum_{M \in \mathcal{M}_n} M$.

Let σ be a finite model. Can relativise the whole theory to models with a labelled embedded copy of σ, get a more general flag algebra $\mathcal{F}_\sigma$. Define an *averaging operator* which takes $A \in \mathcal{F}_\sigma$ and returns $[A]_\sigma \in \mathcal{F}_0$.

Theorem (Razborov) *Cauchy-Schwartz theorem*: If $A \in \mathcal{F}_\sigma$, then $[A \cdot A]_\sigma \geqslant 0$.

Idea: Can prove inequalities of the form $B \geqslant r$ by adding equations of the form $[A_i \cdot A_i]_{\sigma_i} \geqslant 0$ for various σ_i and $A_i \in \mathcal{F}_{\sigma_i}$.

Idea: Using the optimisation technique of *semidefinite programming*, can search efficiently for the best provable value of r in the inequality $B \geqslant r$

Problem: What is the minimum possible density of monochromatic triangles in a 3-colouring of the edges of K_n for large n?

Easy example: Take the 2-colouring of the edges of K_5 with no monochromatic triangle. Blow up each vertex to a monochromatic clique of size n in a third colour. Shows we can't do better than $1/25$ asymptotically.

Theorem (C., Kral, Pfender, Sperfeld, Treglown and Young) $1/25$ is optimal.

Main point: use semidefinite programming to search for a proof of the inequality "red triangle plus blue triangle plus green triangle $\geqslant 1/25$" in the flag algebra for 3-coloured complete graphs.

Idea: Same setting as theory of flag algebras. Associate to a convergent sequence of finite models a set of measurable functions. This serves as a "template" for constructing a random sequence which induces the same Φ.

(Lovasz and Szegedy) Graph limits. Associate to a convergent sequence of finite graphs a symmetric measurable function $L : [0,1]^2 \to [0,1]$.

To use L as a template for a graph on n vertices: Choose $r_i \in [0,1]$ for $i = 1 \ldots n$ uniformly and independently. Then join v_i and v_j with probability $L(r_i, r_j)$.

Instructive example (Elek and Szegedy) Hypergraph limits: The limit object for a convergent sequence of 3-uniform hypergraphs is a function of six variables $x_{\{1\}}, x_{\{2\}}, x_{\{3\}}, x_{\{1,2\}}, x_{\{1,3\}}, x_{\{2,3\}}$ symmetric under the action of S_3.

Elek and Szegedy's proof uses Löb measures (ultraproducts of counting measures on finite sets) and Maharam's theorem. Also use Borel-Cantelli but now need stronger estimates (Chernoff bound) for deviation of random variables from their means.

(Szemeredi) Regularity and Removal lemmas for finite graphs.

Regularity (roughly): a large finite graph can be decomposed into a small number of roughly equal chunks (plus a tiny bad set) such that edges between each pair of chunks strongly resemble edges of some random bipartite graph.

Triangle Removal: For all $\epsilon > 0$ there is $\delta > 0$ such that any large finite graph with at most δn^3 triangles can be rendered triangle-free by removing at most ϵn^2 edges.

Graph limits give a set of “transfer principles” for translating analytic facts about measurable functions into combinatorial facts about large finite graphs. For example the Lebesgue density theorem translates into Removal.

Theorem (Aroskar and C.) There is a theory of limits which works in the same generality as the theory of flag algebras. In particular we can prove general versions of Regularity and Removal for finite models of a universal theory T written in a finite relational language.