Ultrafilter on N and P-subfilters

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Origins

Definition

1. A point p of $\beta X \setminus X$ is a remote point of X if $p \notin \overline{D}$ for all nwd (nowhere dense) $D \subset X$
2. a filter $\mathcal{F}$ on ω is a P-filter if it is countably directed mod finite
3. a filter on ω is nwd if it has no pseudointersection,
4. A subset P of a space X is a P-set if its neighborhood filter is countably directed

Theorem

1. *if compact X can be covered by nwd P-sets, then $\omega \times X$ has no remote points*
2. *every uniform ultrafilter on ω_2 has a nwd P-subfilter and so the space $\omega \times U(\omega_2)$ has no remote points,*
3. *CH implies not every ultrafilter on ω has a nwd P-subfilter*

Definition

1. a tower on ω is a (maximal) $^*\supset$ well-ordered subset of $[\omega]^\omega$ (hence a nwd P-filter)
2. A family $\mathcal{S} \subset \omega^\omega$ is a scale if it is a $<^*$-unbounded and is a $<^*$-well-ordered chain. The minimum cardinality of a scale is $\mathfrak{b}$

Theorem (Balcar-Frankiewicz-Mills, 1980)

If there is a scale $\mathcal{S}$ with $|\mathcal{S}| > \mathfrak{b}$, then every ultrafilter on ω contains a (non-meager) tower.

Theorem (Blass, Blass-Shelah, 1989)

NCF implies every ultrafilter on ω contains a nwd P-subfilter.

Also connections to the (still open) Scarborough-Stone problem

back to scales and nwd P-subfilters

What Balcar, Frankiewicz, and Mills actually showed

Proposition

If $\mathcal{S}$ is a λ-scale that is not cofinal in the $<_{\mathcal{U}}$-ordering on ω^ω, then $\mathcal{U}$ extends a (non-meager) tower of cofinality λ

Which may have led Peter Nyikos to formulate 6 of the 8 possible variants of

$$(\forall\exists\mathcal{S})(\forall\exists\mathcal{U})(\mathcal{S} \text{ is cofinal in } <_{\mathcal{U}}) \text{ and}$$
$$(\forall\exists\mathcal{U})(\forall\exists\mathcal{S})(\mathcal{S} \text{ is cofinal in } <_{\mathcal{U}}) \text{ as Axioms}$$

in particular

"Ax2: $(\exists\mathcal{S})(\forall\mathcal{U})$" and "Ax3: $(\exists\mathcal{U})(\forall\mathcal{S})$"

easily seen that Ax2 implies $\mathfrak{b} = \mathfrak{d}$

and any $\mathcal{U} \not\models$Ax3 extends a non-meager tower

more on Ax3 – or rather $(\exists\mathcal{U})\ \mathcal{U} \models$Ax3

Nyikos asked if Ax2 implies Ax3 (and other questions)

Fact [Nyikos]

Ax3 implies that every $\mathcal{S}$ has cofinality $\mathfrak{b}$.

Theorem

If $\mathcal{U}$ extends a non-meager tower, then (iff) $\mathcal{U} \not\models$Ax3.

Theorem

$\mathfrak{b} = \mathfrak{d}$ *implies there is a $\mathcal{U}$ that does not extend a non-meager tower and so Ax3 holds.*

thus Ax2 (which implies $\mathfrak{b} = \mathfrak{d}$) does imply Ax3

e.g. known in Laver model there is no non-meager tower

does every ultrafilter have a nwd P-subfilter?

Proposition

CH implies there is a $\mathcal{U}$ with no nwd P-subfilter

Question

Does PFA imply that every ultrafilter has a nwd P-subfilter?
(MA + $\neg$CH does not)

Lemma (Shelah)

for a usual proper iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha \in \omega_2\rangle$ in which each $\dot{Q}_\alpha$ does not fill nwd P-filters, P_{ω_2} itself will not fill nwd P-filters
Hence (by reflection) such P_{ω_2} forces that each nwd P-filter has a nwd P-subfilter of character $\aleph_1$

Corollary

If G is P_{ω_2}-generic and $\mathcal{U} \in V[G]$, then we seek a $\mu < \omega_2$ such that $\mathcal{U} \cap V[G_\mu]$ has a nwd P-subfilter.

Scenario

If G is P_{ω_2}-generic and $\mathcal{U} \in V[G]$,

$$\text{and } M \prec H(\theta),\ M^\omega \subset M,\ M \cap \omega_2 = \lambda$$

then $\mathcal{U}_\lambda = \mathcal{U} \cap M[G_\lambda]$ is an ultrafilter but CH holds so it may not contain a P-subfilter in $M[G_\lambda]$

Corollary (for example)

If for stationarily many λ (of cof. ω_1), $\dot{Q}_\lambda$ is Miller forcing, then (for such a λ) $\mathcal{U} \cap V[G_\lambda]$ contains a nwd P-subfilter in $V[G_{\lambda+1}]$

Nyikos: How about the Laver model?

I want to sketch the proof that "for a completely different reason", the following also holds in the Laver model

Theorem

For each non-P-point $\mathcal{U}$ *and suitably reflecting* λ*,*

$\mathcal{U} \cap V[G_{\lambda+1}]$ *has a nwd P-subfilter*

and one of the main ideas was based on analyzing which Laver names $\dot{Y}$ are not in $\mathcal{F}^+$ for non-meager P-filters $\mathcal{F}$

Similar to no rapid ultrafilters and Miller preserves P-points

and the second is to take advantage of $\mathcal{U} \cap V[G_{\lambda+1}]$

For a non-P-point $\mathcal{U}$,
say that a partition $\langle A_n : n \in \omega \rangle$ is a $\mathcal{U}$-partition if for all $f \in \omega^\omega$, $\bigcup\{A_n \setminus f(n) : n \in \omega\}$ is in $\mathcal{U}$.

Lemma

If $\mathcal{U}$ is not a P-point and λ is reflecting, then, for each $\mathcal{U} \cap V[G_\lambda]$-partition $\langle A_n : n \in \omega \rangle$ in $V[G_\lambda]$ and each $f \in V[G_{\lambda+1}] \cap \omega^\omega$ (for example, the Laver real),

$$\bigcup\{A_n \setminus f(n) : n \in \omega\} \text{ is in } \mathcal{U} \ (\cap V[G_{\lambda+1}])$$

Now my plan is to build a Laver name of a (dual) dense P-ideal $\mathcal{Y}$ such that $\mathcal{Y}$ is disjoint from $(\mathcal{U} \cap V[G_{\lambda+1}])^+$.

building the Laver name $\dot{\mathcal{Y}}$

Definition

For a Laver condition $T \subset \omega^{<\omega}$, $t \in T$,
$T_t = \{s \in T : s \cup t \in \omega^{<\omega}\}$ and, for $m \in \omega$,
$T_{t,m} = \{s \in T_t : s \subseteq t \text{ or } s(|t|) > m\}$

and then for a name $\dot{Y}$,
say that $T_t \Vdash_w k \in\notin \dot{Y}$ if $T_{t,m} \Vdash k \in\notin \dot{Y}$ for some m.

Definition

A Laver condition $T \subset \omega^{<\omega}$ is $\dot{Y}$-trimmed if for (a front of) $t \in T$ and sets $\dot{Y}_{T,t} \subset \omega$, such that

1. $T_{t^\frown n}$ decides the value of $\dot{Y} \cap n$,
2. $T_t \Vdash_w k \in\notin \dot{Y}$ iff $k \in \notin \dot{Y}_{T,t}$

Lemma

For any countable set $\{\dot{Y}_n : n \in \omega\}$, the set of T that are $\dot{Y}_n$-trimmed, for each n, is dense.

for a $t \in \omega^\omega$, let t^- denote the immediate predecessor.

Definition

$\dot{Y}$ is T-thin if T is $\dot{Y}$-prepared and $\dot{Y}_{T,t}$ is finite for all $t \in T$.
actually only above a front
and super T-thin if $\{\dot{Y}_{T,t} \setminus \dot{Y}_{T,t^-} : t \in T\}$ are pairwise disjoint

Lemma

If $T \Vdash \dot{Y} \in \mathcal{U}^+$ *and* $\dot{Y}$ *is super* T*-thin,*

then the sequence $\{A_t : t \in T\}$ *is a* $\mathcal{U}$*-partition*

where $A_t = \bigcup\{\dot{Y}_{T,t^\frown k} \setminus \dot{Y}_{T,t} : t^\frown k \in T\}$

Lemma

If $\dot{Y}$ *is* T*-thin, then there are* $T' <_0 T$ *such that*
$\dot{Y}$ *is super* T'*-thin*

and the name $\dot{\mathcal{Y}} = \{(T, \dot{Y}) : \dot{Y}$ *is* T*-thin*$\}$ *WORKS!*

Now for the Cohen model and square

Theorem (Assume $\square_{\omega_1}$ and CH)

Forcing with $Fn(\omega_2, 2)$ creates a model in which there is a $\mathcal{U}$ that does not have a nwd P-subfilter (of character $\aleph_1$).

(and adding ω_2 Random reals will have P-points)

Proof.

Let $\mathcal{A}$ vary over $Fn(\omega_2, 2)$-names $\{\dot{a}^{\mathcal{A}}_{\alpha} : \alpha \in \omega_1\}$ that are forced to be $\subset^*$-increasing dense (P-)ideals.

For each $\mathcal{A}$, we must choose $\delta(\mathcal{A})$ so that

$$1 \Vdash a^{\mathcal{A}_0}_{\delta(\mathcal{A}_0)} \cap \cdots \cap a^{\mathcal{A}_\ell}_{\delta(\mathcal{A}_\ell)} \neq \emptyset$$

for any such $\mathcal{A}_i$ ($i \leq \ell$); which extends to $\mathcal{U}$. □

Let $\{C_\gamma : \gamma \in \omega_2\}$ be a $\square$-sequence

1. $C_\gamma \subset \gamma$ is a cub and $|C_\gamma| \leq |\,cf(\gamma)|$,
2. $\gamma \in C'_\eta$ implies $C_\gamma = C_\eta \cap \gamma$.

Definition

For a countable $M \prec H(\aleph_2)$,
let $\delta_M = M \cap \omega_1$ and $\gamma_M = \sup(M \cap \omega_2)$

Fix $\vec{g} = \{g_\xi : \omega_1 \leq \xi \in \omega_2\}$ with g_ξ mapping ω_1 onto $[\xi]^{\aleph_0}$

Definition

$M \in \mathcal{M}$ providing $M \prec (H(\aleph_2), \square, \vec{g})$ is countable and
$o.t.(C_{\gamma_M}) = \delta_M$. ($\mathcal{M}$ is stationary)

Cohen model and elementary submodels

Here is the main ingredient (I_1 is the support of a name)

Lemma

Let $M_1, M_2 \in \mathcal{M}$ with $\delta_1 = M_1 \cap \omega_1 \leq \delta_2$.
For any $I_1 \in M_1 \cap [\omega_2]^{\aleph_0}$, $I_1 \cap M_2$ is in $M_1 \cap M_2$.

Proof.

Let $\beta = \sup(I_1 \cap M_2)$; which is in M_1 and $o.t.(C_\beta) < \delta_1 \leq \delta_2$.

Hence $\beta < \gamma_{M_2}$ and let $\lambda = \min(M_2 \cap \omega_2 \setminus \beta)$. Clearly $\beta \in C'_\lambda$.
Then $\beta \in M_2$ since $M_2 \models (\exists \beta \in C'_\lambda)\ o.t(C_\lambda \cap \beta) = o.t.(C_\beta)$

Now pick $\alpha < \delta_1 \leq \delta_2$ so that $g_\beta(\alpha) = I_1 \cap \beta \in M_2$. □

Corollary

Let $M_1, \ldots, M_n$ *be in* $\mathcal{M}$ *enumerated so that* $\delta_i \leq \delta_{i+1}$, *where* $\delta_i = M_i \cap \omega_1$. *For each* $1 \leq i \leq n$, *let* $I_i \in M_i \cap [\omega_2]^{\aleph_0}$, *s.t., for convenience,* $I_i \cap M_{i+1} \subset I_{i+1}$

For each $1 \leq i \leq n$, *the boolean algebra generated by* $\{I_1 \cap M_i, \ldots, I_{i-1} \cap M_i\} \cup \{I_i \cap M_j : i \leq j \leq n\}$ *is in* M_i.

Corollary

If M_1, M_2 *are as above,* $\dot{a}_1 \in M_1$ *with* $1 \Vdash \dot{a}_1 \in [\omega]^{\aleph_0}$ *and*

if $\mathcal{A}_2 \in M_2$*, then there is* $\dot{a}_2 \in \mathcal{A}_2 \cap M_2$
such that $1 \Vdash \dot{a}_1 \cap \dot{a}_2 \in [\omega]^{\aleph_0}$

Proof.

Let I_1 be the support of $\dot{a}_1$ and choose $\mu_2 \in M_2 \cap \omega_2$ such that (1) $I_1 \cap M_2 \subset \mu_2$ and (2) $\bigcup\{supp(\dot{a}) : \dot{a} \in \mathcal{A}_2\} \subset \mu_2$
Let ψ be a 1-to-1 function such that $\psi \restriction (I_1 \cap M_2)$ is the identity, and $\psi \restriction (I_1 \setminus M_2)$ is canonical order-preserving into $[\mu_2, o.t.(I_1)]$

then $\psi(\dot{a}_1) \in M_2$

and there is $\dot{a}_2 \in \mathcal{A}_2$ with $1 \Vdash \psi(\dot{a}_1) \cap \dot{a}_2 \in [\omega]^{\aleph_0}$

hence, by supports, $1 \Vdash \dot{a}_1 \cap \dot{a}_2 \in [\omega]^{\aleph_0}$ □

Similarly

Lemma

Let $\mathcal{A}_i \in M_i$ with $\{M_i : 1 \leq i \leq n\}$ as above, then, by induction on i, there is $\dot{a}_i \in \mathcal{A}_i \cap M_i$ (and $I_i = supp(\dot{a}_i)$) such that

$$1 \Vdash \dot{a}_1 \cap \cdots \cap \dot{a}_i \text{ is infinite}$$

Corollary

For each $\mathcal{A}$, choose $\mathcal{A} \in M_{\mathcal{A}} \in \mathcal{M}$, let $\delta(\mathcal{A}) = M_{\mathcal{A}} \cap \omega_1$

and recall notation $\mathcal{A} = \{\dot{a}^{\mathcal{A}}_{\alpha} : \alpha \in \omega_1\}$ is an increasing P-ideal,

$1 \Vdash \{\dot{a}^{\mathcal{A}}_{\delta(\mathcal{A})} : \mathcal{A}$ a suitable Cohen name$\}$ extends to an ultrafilter.