The Nonexistence of Universal Metric Flows

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Introduction

Let T be a topological (semi-) group.

A *T-flow* is a compact space X, the *phase space*, together with a continuous map

$$T \times X \to X; (\tau, x) \mapsto \tau x$$

satisfying

$$(\tau_1 \tau_2)x = \tau_1(\tau_2 x)$$

for all $\tau_1, \tau_2 \in T$ and all $x \in X$.

A map $f : X \to Y$ between two T-flows is *equivariant* if for all $x \in X$ and all $\tau \in T$ we have

$$f(\tau x) = \tau f(x).$$

A T-flow Y is a *factor* of a T-flow X if there is continuous equivariant function from X onto Y.

If $\mathcal{C}$ is a class of T-flows, then a flow $X \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $Y \in \mathcal{C}$ is a factor of X.

We turn to a more specific situation.
For a compact space X, every continuous map $f : X \to X$ gives rise to a continuous action of the semigroup $\mathbb{N}$ on X:

$$\mathbb{N} \times X \to X; (n, x) \mapsto f^n(x)$$

If f is a homeomorphism, it induces a $\mathbb{Z}$-action on X in the same way.

Definition

If X is a compact space, $f : X \to X$ is continuous and $x \in X$, then the ω-limit set of x is the set

$$\omega(x) = \bigcap_{n \geq 0} \mathrm{cl}\{f^m(x) : m \geq n\}.$$

The ω-limit set of a point x is closed under f and hence a flow in itself.
An *abstract ω-limit set* is an $\mathbb{N}$-, respectively $\mathbb{Z}$-flow that is isomorphic to an ω-limit set of a point.

Question (Will Brian)

Is there a universal abstract metric ω-limit set?

There is a universal abstract ω-limit set, i.e., the answer to this question is positive if we drop the metrizability assumption.

Consider the Čech-Stone compactification $\beta\mathbb{N}$ and extend the shift $n \mapsto n + 1$ to all of $\beta\mathbb{N}$. Remove the isolated points and obtain the dynamical system $\beta\mathbb{N} \setminus \mathbb{N}$, the ω-limit set of 0. This is universal in the class of all abstract ω-limit sets, but not metrizable.

A T-flow X is *minimal* if no proper closed subsets of X is closed under the action of T.
It is well known that there are universal minimal flows, but for $T = \mathbb{N}$ or $T = \mathbb{Z}$ these are not metrizable.
This brings up the question whether there are universal objects in the class of metric minimal flows.

The answer is negative, and this follows from Furstenberg's structure theorem for minimal distal metric flows and a result of Beleznay and Foreman.

Furstenberg's theorem allows it to assign an ordinal rank to every minimal distal metric flow that cannot increase when taking factors, the *distal height* of the flow. The the distal height of a minimal distal metric flow turns out to be countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow.
But every minimal metric flow has a maximal distal factor that is also minimal.
Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Anderson's theorem and Stone duality

Theorem (Anderson)

Let $T = \mathbb{N}$ or $T = \mathbb{Z}$. Then every metric T-flow X is a factor of a T-flow with the phase space $2^{\mathbb{N}}$. If X is minimal, then the flow on $2^{\mathbb{N}}$ can be chosen to be minimal as well.

There is an analogue of Anderson's theorem for abstract ω-limit sets.

If X is 0-dimensional and $f : X \to X$ is continuous, then we can consider the Boolean algebra $\mathrm{Clop}(X)$ of clopen subsets of X together with the endomorphism f^* defined by $f^*(a) = f^{-1}[a]$.

Every 0-dimensional T-flow X gives rise to a Boolean algebra $\mathrm{Clop}(X)$ on which T acts by endomorphisms, a *Boolean algebraic T-flow*.

If X and Y are T-flows and $h : X \to Y$ is continuous and equivariant, then $h^* : \mathrm{Clop}(Y) \to \mathrm{Clop}(X)$ defined by $h^*(a) = h^{-1}[a]$ is an equivariant homomorphism.

h is onto iff h^* is 1-1. Hence, by Anderson's theorem, if we are interested in universal objects, instead of looking at factors of metric flows, we can study embeddings of countable Boolean algebraic flows.

Definition

Let $T \in \{\mathbb{N}, \mathbb{Z}\}$. If A is a Ba T-flow and $a \in A$, then by $\langle a \rangle_T$ we denote the smallest subalgebra B of A such that $a \in B$ and B is closed under the action by T. The Boolean algebra $\langle a \rangle_T$ is the subalgebra of A generated by the T-orbit of a.

Definition

Given two Ba T-flows A and B and elements $a \in A$ and $b \in B$, we call the pairs (A, a) and (B, b) *isomorphic* if there is an isomorphism between the T-flows A and B that maps a to b.

Definition

Given a Ba T-flow A and $a \in A$, the *type* of a is the isomorphism type of the pair $(\langle a \rangle_T, a)$.

If A is a Ba T-flow and $I \subseteq A$ is an ideal that is closed under the action, then T acts on the quotient A/I.

On the other hand, the kernel of an equivariant homomorphism from a Ba T-flow A to a Ba T-flow B is an ideal that is closed under the action.

Definition

For $T \in \{\mathbb{N}, \mathbb{Z}\}$ let $\mathrm{Fr}(T)$ be the free Boolean algebra over the set $\{g_n : n \in T\}$ of generators. We assume that the g_n are pairwise distinct. Let $s_T : \mathrm{Fr}(T) \to \mathrm{Fr}(T)$ be the Boolean homomorphism extending the map $g_n \mapsto g_{n+1}$. This induces a T-action on $\mathrm{Fr}(T)$.

Lemma

Let A be a BA T-flow for $T = \mathbb{N}$ or $T = \mathbb{Z}$ and let $a \in A$. Then there is a unique Boolean homomorphism $\pi : \mathrm{Fr}(T) \to A$ such that $\pi(g_0) = a$ and $\pi(s_G(b)) = f(\pi(b))$ for all $b \in \mathrm{Fr}(T)$.

Lemma

Let A and B be Ba T-flows for $T = \mathbb{N}$ or $T = \mathbb{Z}$, $a \in A$, and $b \in B$. Suppose that $A = \langle a \rangle_T$ and $B = \langle b \rangle_T$.
Let $\pi_A : \mathrm{Fr}(T) \to A$ and $\pi_B : \mathrm{Fr}(T) \to B$ be the unique equivariant homomorphims with $\pi_A(g_0) = a$ and $\pi_B(g_0) = b$.
Then a and b have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are identical.

Symbolic dynamics

Definition

Let $T = \mathbb{N}$ or $T = \mathbb{Z}$. On the space $\{0,1\}^T$ we consider the shift $S_T : \{0,1\}^T \to \{0,1\}^T$ which is defined by letting $S_T(x) : G \to \{0,1\}$ be the map satisfying $S(x)(n) = x(n+1)$ for all $n \in T$. Clearly, $S_{\mathbb{Z}} : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ is a homeomorphism and $S_{\mathbb{N}} : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is a continuous map.

Note that the shift S_T on $\{0,1\}^T$ is (isomorphic to) the Stone dual of the shift s_T on $\mathrm{Fr}(T)$.

The goal is now to construct many types of 1-generated Ba
T-flows by looking at *subshifts* of S_T.

Definition

Let $T = \mathbb{N}$ or $T = \mathbb{Z}$. A *Sturmian word* is a word $x \in \{0,1\}^T$ such that there are two real numbers, the *slope* α and the *intercept* ρ, with $\alpha \in [0,1)$ irrational such that for all $i \in T$ we have

$$x(i) = 1 \quad \Leftrightarrow \quad (\rho + i \cdot \alpha) \mod 1 \in [0, \alpha).$$

It is well known that the orbit closure $C_x = \mathrm{cl}\{s_T^n(x) : n \in T\}$ of a Sturmian word with the restriction of the shift is a minimal T-flow. Also, the slope α can be computed from every element of the orbit closure of x.

It follows that for different irrational numbers $\alpha, \beta \in [0,1)$, Sturmian words of slope α and β have different (even disjoint) orbit closures.
We call the orbit closure of a Sturmian word together with the restriction of S_X a *Sturmian subshift*. A Sturmian subshift is a X-flow.

Given a Sturmian subshift $(X, S_T \restriction X)$, we denote the common slope of all Sturmian words that generate X by $\alpha(X)$.

Lemma

Let $(X, S_T \restriction X)$ be a Sturmian subshift and let $p : \mathrm{Fr}(T) \to \mathrm{clop}(X)$ be the homomorphism dual to the embedding of X into $\{0,1\}^T$. Then $\langle p(g_0)\rangle_T = \mathrm{clop}(X)$ and the type of $p(g_0)$ determines $\alpha(X)$.

Theorem

Let $G = \mathbb{N}$ or $T = \mathbb{Z}$. Then there is no metric T-flow that has all Sturmian subshifts as factors.

Corollary

Let $T = \mathbb{N}$ or $T = \mathbb{Z}$. The following classes of T-flows contain no universal elements:

1. *Metric T-flows*
2. *Metric minimal T-flows*
3. *Metric abstract ω-limit sets*

Thank you for your attention!