Strong Witnesses for Cardinals in Cichoń's Diagram

Martin Goldstern
(joint work with Jakob Kellner and Saharon Shelah)

Cichoń's Maximum, arXiv:1708.03691

Technische Universität Wien

Eilat, April 2018

Outline

Background

Main Theorem

Linear witnesses and cone witnesses

Large cardinals

Cone witnesses

Construction

Outline

Background

Main Theorem

Linear witnesses and cone witnesses

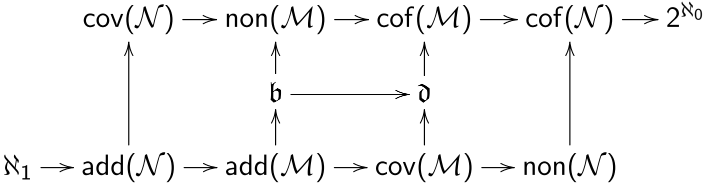
Large cardinals

Cone witnesses

Construction

Cichoń's Diagram

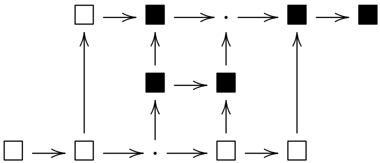
$\mathcal{M}$ = the ideal of meager subsets of $\mathbb{R}$.
$\mathcal{N}$ = the ideal of Lebesgue null sets of $\mathbb{R}$.



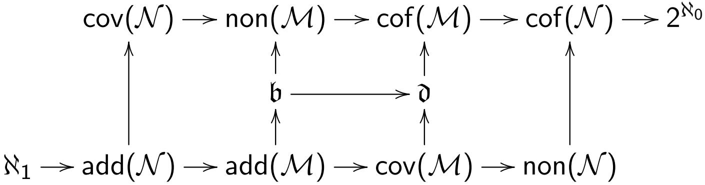
Are these cardinals different?

Examples

- CH $\Leftrightarrow$ all these cardinals are equal.
- MA + $\neg$CH $\Rightarrow$ 2 values: $\aleph_1 < \text{add}(\mathcal{N}) = 2^{\aleph_0}$.
- Many other consistency results for 2 values. e.g.



- Many consistency results for more than 2 values.



In ZFC:

$$\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$$
$$\mathrm{cof}(\mathcal{M}) = \max(\mathrm{non}(\mathcal{M}), \mathfrak{d})$$

Outline

Background

Main Theorem

Linear witnesses and cone witnesses

Large cardinals

Cone witnesses

Construction

Main theorem

(G-Kellner-Shelah 2017, `arXiv:1708.03691`)

Starting from a universe with 4 strongly compact cardinals, we construct a universe in which 10 values $\aleph_1 = \lambda_0 < \cdots < \lambda_9 = 2^{\aleph_0}$ appear in Cichon's diagram:

$$
\begin{array}{ccccccccc}
 & & \lambda_2 & \rightarrow & \lambda_4 & \longrightarrow \cdot \longrightarrow & \lambda_8 & \rightarrow & \lambda_9 \\
 & & \uparrow & & \uparrow & \quad\quad\quad\;\; \uparrow & \uparrow & & \\
 & & & & \lambda_3 & \rightarrow \quad \lambda_6 & & & \\
 & & & & \uparrow & \quad\quad\quad\;\; \uparrow & & & \\
\lambda_0 & \rightarrow & \lambda_1 & \longrightarrow & \cdot & \longrightarrow \lambda_5 \rightarrow & \lambda_7 & &
\end{array}
$$

Diego Mejía: Need only 3 strongly compact.
Moti Gitik: Less is needed: only ∼superstrongs (below subcompacts) instead of strongly compacts.

Outline

Background

Main Theorem

Linear witnesses and cone witnesses

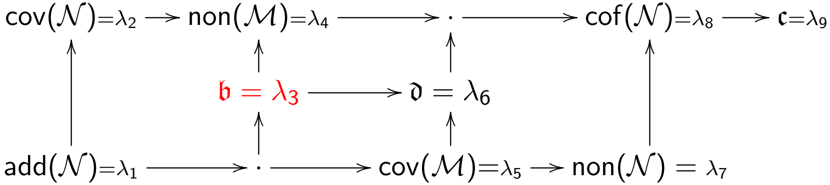
Large cardinals

Cone witnesses

Construction

Witnesses

We start with a model of GCH and "carefully" increase the cardinals in Cichoń's diagram through forcing.



When dealing with (for example) $\mathfrak{b}$ and $\mathfrak{d}$, we have to make sure that in the resulting model we have
$\mathfrak{b} \geq \lambda_3$, $\mathfrak{b} \leq \lambda_3$, $\mathfrak{d} \geq \lambda_6$, $\mathfrak{d} \leq \lambda_6$.
We will write λ_b, λ_d instead of λ_3, λ_6.

Witnesses

- A *witness* for $\mathfrak{b} \leq \lambda$ is an unbounded family of functions (of size λ), i.e., a family $(f_i : i < \lambda) \in (\omega^\omega)^\lambda$ of functions $f_i \in \omega^\omega$ such that $\forall g \in \omega^\omega \; \exists i < \lambda : f_i \not\leq^* g$.

Similar definitions can be made for some other characteristics in the diagram.
For example, a witness for $\mathrm{non}(\mathcal{M}) \leq \lambda$ is a family $(x_i : i < \lambda)$ of reals which is not meager (equivalently: for every code y of a meager Borel set M_y there is $i < \lambda$ such that $x_i \notin M_y$).
(For simplicity we will concentrate on $\mathfrak{b}$ and $\mathfrak{d}$.)

Witnesses and linear witnesses

- $(f_i : i < \lambda) \in (\omega^\omega)^\lambda$ is a *witness* for $\mathfrak{b} \leq \lambda$ if: $\forall g \in \omega^\omega\ \exists i < \lambda : f_i \not\leq^* g$.
- A family $(f_i : i < \lambda) \in (\omega^\omega)^\lambda$ is a *linear witness* for $\mathfrak{b} \leq \lambda$ if:

$$\forall g \in \omega^\omega\ \forall^\infty i < \lambda : f_i \not\leq^* g.$$

(Here, $\forall^\infty i \cdots$ means $\exists i_0\ \forall i > i_0 : \cdots$)
$\mathrm{LCU}(\lambda) :=$ "there exists a linear witness of length λ".

Remark

Clearly $\mathrm{LCU}(\lambda) \Leftrightarrow \mathrm{LCU}(cf(\lambda))$.
If λ is regular, then: $\mathrm{LCU}(\lambda)$ *implies not only* $\mathfrak{b} \leq \lambda$ *but also* $\lambda \leq \mathfrak{d}$.

Linear witnesses, an example

$\vec{f} \in (\omega^\omega)$ is a *linear witness* if: $\forall g \in \omega^\omega\ \forall^\infty i < \lambda : f_i \not\leq g$.

Example

Let $(P_\alpha, Q_\alpha : \alpha < \mu)$ be a finite support iteration where each Q_α is just Cohen forcing $\omega^{<\omega}$, adding a Cohen real g_α.
Then $(g_\alpha : \alpha < \mu)$ is a linear witness (easy), but also:

- For every $\lambda \in [\aleph_1, \mu]$: $(g_\alpha : \alpha < \lambda)$ is a linear witness.

So in V^{P_μ} we have $\mathrm{LCU}(\lambda)$ for all (regular) $\lambda \in [\aleph_1, \mu]$.
We will abbreviate this as $P \Vdash \mathrm{LCU}([\aleph_1, \mu])$ or $\mathrm{LCU}(P, [\aleph_1, \mu])$.

Keeping $\mathfrak{b} \leq \lambda_b$ in ultrapowers

$\vec{f} \in (\omega^\omega)$ is a *linear witness* if: $\forall g \in \omega^\omega\ \forall^\infty i < \lambda : f_i \not\leq^* g$.
LCU(λ) says that such $\vec{f}$ exists.

Lemma

If $j : V \to M$ is elementary with $cp(j) = \kappa$ with many nice properties (wait. . .), and $P \Vdash \mathrm{LCU}(\lambda_b)$, then $j(P) \Vdash \mathrm{LCU}(j(\lambda_b))$. If $\lambda_b \neq \kappa$, then also $j(P) \Vdash \mathrm{LCU}(\lambda_b)$.

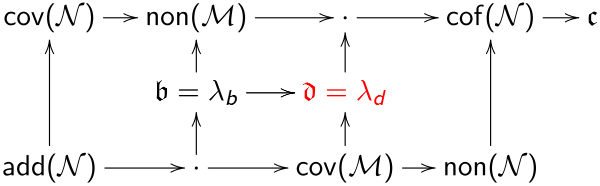
Proof.

Assume that $P \Vdash \underset{\sim}{\vec{f}}$ is a linear witness.
Then both $j(\underset{\sim}{\vec{f}})$ and $j``\underset{\sim}{\vec{f}} = (j(\underset{\sim}{f_i}) : i < \lambda_b)$ are linear witnesses. (As $j``\lambda_b$ is cofinal in λ_b.) □

This will allow us to keep $\mathfrak{b} \leq \lambda_b$.

Increasing $\mathfrak{d}$ to be $\geq \lambda_d$

Our goal:



Lemma

Let $j : V \to M$ with $cf(j(\kappa)) = \lambda_d$, and $\lambda_b < \kappa < \lambda_d$.
Assume that $\mathrm{LCU}(P, \kappa)$*).*
Then $\mathrm{LCU}(j(P), \lambda_d)$*, hence* $j(P) \Vdash \mathfrak{d} \geq \lambda_d$*.*

Outline

Background

Main Theorem

Linear witnesses and cone witnesses

Large cardinals

Cone witnesses

Construction

Large cardinals

Using a strongly compact cardinal κ, and $\theta > \kappa$ regular, get a κ^+-cc κ-complete Boolean algebra B and a κ-complete ultrafilter on B such that the ultrapower embedding $j : V \to M$ has the following properties:

- M is $< \kappa$-closed
- each element of M is a "mixture" of κ elements of V (along a maximal antichain)
- (For regular $\lambda > \kappa$, every $<\lambda$-directed partial order S: $j``S$ is cofinal in $j(S)$.)
- $cf(j(\kappa)) = \theta$, $j(\kappa) < \theta^+$

[Gitik: Instead use an extender ultrapower with large regular $j(\kappa)$. Every $x \in M$ is of the form $j(f)(b)$, $f : \kappa^{|b|} \to V$, b finite.]

Outline

Background

Main Theorem

Linear witnesses and cone witnesses

Large cardinals

Cone witnesses

Construction

Ensuring $\mathfrak{d} \leq \lambda_d$ and $\mathfrak{b} \leq \lambda_b$; cone witnesses

$\mathcal{G} = (g_j : j < \lambda)$ is a witness for $\mathfrak{d} \leq \lambda$ iff $\mathcal{G}$ dominates:

$$\forall f \in \omega^\omega : \exists j \; f \leq^* g_j$$

Let λ, μ be regular uncountable.
$\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(P, \lambda, \mu)$ means that there is a (λ, μ)-cone witness, that is: $<\lambda$-directed partial order $(S, \leq)$ of size μ in V together with a family $(g_s : s \in S)$ of P-names of functions $\underset{\sim}{g}_s \in \omega^\omega$ such that

$$P \Vdash \forall f \in \omega^\omega \; \forall^\infty s \in S : f \leq^* \underset{\sim}{g}_s$$

As above, $\forall^\infty s \in S$ means "eventually", i.e., $\exists s_0 \in S \; \forall s > s_0 \ldots$

Lemma

$\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(P, \lambda, \mu) \Rightarrow P \Vdash \mathfrak{b} \geq \lambda$ *and* $\mathfrak{d} \leq \mu$.

Cone witnesses, example ($\mathfrak{b} \geq \lambda$)

Example

Let $S \subseteq \delta$, and let $(w_\alpha : \alpha \in S)$ be a family of sets which is cofinal in $[\delta]^{<\lambda}$, with $w_\alpha \subseteq \alpha$ for all α.

We order S by $\alpha \sqsubseteq \beta \Leftrightarrow w_\alpha \subseteq w_\beta$. This partial order is $<\lambda$-directed.

Let $(P_\alpha, Q_\alpha : \alpha < \delta)$ be a "standard" finite support ccc iteration designed to make $\mathfrak{b} \geq \lambda$, based on $(w_\alpha : \alpha \in S) \subseteq [\delta]^{<\lambda}$, $S \subseteq \delta$ (each Q_α introduces a dominating g_α over $V^{\vec{P} \restriction w_\alpha}$).

Then in V^{P_δ}, the sequence $(g_\alpha : \alpha \in S)$ is a $(\lambda, |S|)$-cone witness.

Proof.

Check that $P \Vdash \forall f \, \forall^\infty \alpha \in S : f \leq^* g_\alpha$. □

So we have $\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(\lambda, |S|)$, so $\mathfrak{b} \geq \lambda$, and $\mathfrak{d} \leq |S|$.

Cone witnesses and ultrapowers

COB(P, λ, μ) says: "there is a $<\lambda$-directed partial order S of size μ, such that P forces an 'eventually dominating' family $\vec{g} = (\underset{\sim}{g}_s : s \in S)$:

$$\forall f \; \exists s_0 \in S \; \forall s \geq s_0 : f \leq^* g_s$$

If $j : V \to M$ is sufficiently nice, then COB(P, λ, μ) implies:

- If $\lambda < \kappa$, then COB$(j(P), \lambda, |j(\mu)|)$ (witnessed by $j(S)$ and $j(\vec{g})$)
- If $\lambda > \kappa$, then COB$(j(P), \lambda, \mu)$ (witnessed by $j``S$ and $j``\vec{g}$) (witnessed by $j(S)$ and $j(\vec{g})$)

Hence $j(P)$ forces $\mathfrak{b} \geq \lambda$ and $\mathfrak{d} \leq \mu$.

Outline

Background

Main Theorem

Linear witnesses and cone witnesses

Large cardinals

Cone witnesses

Construction

$$\kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_8 < \lambda_9$$