

THE HAAR MEASURE PROBLEM

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ABSTRACT. An old problem asks whether every compact group has a Haar-nonmeasurable subgroup. A series of earlier results reduce the problem to infinite metrizable profinite groups. We provide a positive answer, assuming a weak, potentially provable, consequence of the Continuum Hypothesis. We also establish the dual, Baire category analogue of this result.

1. INTRODUCTION

Every infinite compact group has a unique translation-invariant probability measure, its *Haar measure*. Vitali sets (complete sets of coset representatives) with respect to a countably infinite subgroup show that such groups have nonmeasurable subsets. We consider the following old problem.

Haar Measure Problem. *Does every infinite compact group have a nonmeasurable subgroup?*

The Haar Measure Problem dates back at least to 1963, when Hewitt and Ross gave a positive answer for abelian groups [5, Section 16.13(d)]. It was explicitly formulated in a paper of Saeki and Stromberg [8]. The problem remains open despite substantial efforts [4, 3, and references therein].

Hernández, Hofmann, and Morris proved that if all subgroups of an infinite compact group are measurable, then the group must be profinite and metrizable [4, Corollary 3.3]. Building on that, Brian and Mislove proved that a positive answer to the Haar Measure Problem is *consistent* (relative to the usual axioms of set theory) [3, Theorem 2.5]. We repeat their argument, for its elegant simplicity, and since this result will take care of the easier case of our main theorem: Let G be an infinite, metrizable profinite group. As a measure space, the group G is isomorphic to the Cantor space with the Lebesgue measure. Let \mathfrak{c} denote the cardinality of the continuum. Consistently, there is in the Cantor space, and thus in G , a nonnull set A of cardinality smaller than \mathfrak{c} . The subgroup of G generated

2010 *Mathematics Subject Classification.* 28C10, 28A05, 22C05, 03E17.

Key words and phrases. Haar measurable, Baire property, profinite group, compact group, closed measure zero.

by A is nonnull, and its cardinality is smaller than \mathfrak{c} . Since sets of positive measure have cardinality \mathfrak{c} , the group generated by A is nonmeasurable.

Brian and Mislove's observation can be viewed as a solution of the Haar Measure Problem under the hypothesis that there is a nonnull set of cardinality smaller than \mathfrak{c} . This hypothesis violates the Continuum Hypothesis. We will show that the Continuum Hypothesis also implies a positive solution. Moreover, for our proof we only assume a weak consequence of the Continuum Hypothesis, which is provable for some groups, and may turn out provable for all groups. A proof of our hypothesis, if found, would settle the Haar Measure Problem.

2. THE MAIN THEOREM

Throughout this section, we fix an arbitrary infinite metrizable profinite group G , and let μ be its Haar probability measure. For each natural number n , the Haar probability measure on the group G^n is the product measure, which is also denoted μ .

Let H be a subgroup of G , and $X = \{x_1, x_2, \dots\}$ be a countable set of variables. The set of all words in the alphabet $H \cup \{x_1^{\pm 1}, x_2^{\pm 1}, \dots\}$ is denoted $H[X]$. Each word $w \in H[X]$ depends on finitely many parameters from the subgroup H , and finitely many variables; let $|w|$ denote the number of variables in w . We view the word w as a continuous function from $G^{|w|}$ to G defined by substituting the group elements for the variables.

Definition 1. Let e be the identity element of the group G . A *Markov set* is a set of the form $w^{-1}(e)$ for $w \in G[X] \setminus G$.

The Markov sets were studied by Markov, as the sets that are closed in all group topologies on G .

Lemma 2. For each element $b \in G$, and each word $w \in G[X] \setminus G$, the set $w^{-1}(b)$ is Markov.

Proof. Consider the word wb^{-1} . □

For a natural number $n \geq 2$, a set $A \subseteq G^n$, and an element $g \in G$, we define

$$A_g := \{ h \in G^{n-1} : (h, g) \in A \},$$

the fiber of the set A over the point g in the group G^{n-1} .

Since Markov sets are closed (and thus measurable), so are their fibers.

Definition 3. A *Markov null set* is a Markov subset of some finite power of the group G , that is also null with respect to the Haar measure μ . A set N is *Fubini–Markov* if either of the following two cases holds:

- (1) The set N is a Markov null subset of G .

- (2) There are a natural number $n \geq 2$ and a Markov null set $A \subseteq G^n$ such that $N = \{g \in G : \mu(A_g) > 0\}$.

While Markov sets may be subsets of an arbitrary power of G , Fubini–Markov sets are always subsets of G . By the Fubini Theorem, we have the following observation.

Lemma 4. *Every Fubini–Markov set is null.* □

We define a cardinal invariant of the group G .

Definition 5. The *Fubini–Markov number* of G , denoted $\mathfrak{fm}(G)$, is the minimal number of Fubini–Markov sets in G whose union has full measure.

Since a countable union of null sets is null, the Fubini–Markov number of a group is necessarily uncountable.

Example 6. For the Cantor group, we have $\mathfrak{fm}(\mathbb{Z}_2^{\mathbb{N}}) = \mathfrak{c}$. Indeed, let G be an *abelian* infinite metrizable profinite group, and N be a Fubini–Markov set. We consider the two cases in the definition.

Assume that $N = w^{-1}(0) \subseteq G$ for some one-variable word $w \in G[X] \setminus G$. Since the group G is abelian, we have $w(x) = x + a$ for some $a \in G$. Then $w(x) = 0$ if and only if $x = -a$, and thus N is a singleton.

Next, for a natural number $n \geq 2$, let $w^{-1}(0) \subseteq G^n$ be a Markov null set, where $w \in G[X] \setminus G$ and $|w| = n$. Since the group G is abelian, we have $w := x_1 + \cdots + x_n + a$, for some $a \in G$. Then, for each $g \in G$, we have

$$(w^{-1}(0))_g = \{(h_1, \dots, h_{n-1}) : h_1 + \cdots + h_{n-1} + g + a = 0\},$$

and thus $(w^{-1}(0))_g$ is a Lipschitz image of the null set $G^{n-2} \times \{0\}$. It follows that $\mu((w^{-1}(0))_g) = 0$ for all $g \in G$, and we have, in the definition, $N = \emptyset$.

A union of fewer than \mathfrak{c} sets that are at most singletons cannot cover a full measure set.

We arrive at our main theorem. Let \mathcal{N} be the ideal of null sets in the Cantor space, and $\text{non}(\mathcal{N})$ be the minimal cardinality of a nonnull subset of the Cantor space. We settle the Haar Measure Problem for groups G with $\text{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$. By Lemma 4, the Continuum Hypothesis implies $\text{non}(\mathcal{N}) = \mathfrak{fm}(G)$. Example 6 shows that for some groups our hypothesis is provable. We do not know whether it is provable for all infinite metrizable profinite groups G . The numbers $\mathfrak{fm}(G)$ are provably larger than some classical cardinal invariants of the continuum; we will return to this in Section 3.

Theorem 7. *Let G be an infinite metrizable profinite group with $\text{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$. Then G has a Haar-nonmeasurable subgroup.*

Proof. If $\text{non}(\mathcal{N}) < \mathfrak{c}$, then the Brian–Mislove argument applies, namely, every nonnull set of cardinality $\text{non}(\mathcal{N})$ generates a nonmeasurable subgroup of G ; see Section 1 for the details. We may thus assume that $\text{non}(\mathcal{N}) = \mathfrak{c}$. By our hypothesis, we have $\mathfrak{fm}(G) = \mathfrak{c}$.

Let $\{N_\alpha : \alpha < \mathfrak{c}\}$ be the family of G_δ null sets. Every null set is contained in some N_α . We define a transfinite, increasing chain of subgroups H_α of G for $\alpha < \mathfrak{c}$. Let H_0 be a countable dense subgroup of G . Let $w \in H_0[X] \setminus H_0$. For distinct elements $b_1, b_2 \in G$, the sets $w^{-1}(b_1)$ and $w^{-1}(b_2)$ are disjoint. Since Markov sets are closed (and thus measurable), the set

$$P_w := \{b \in G : \mu(w^{-1}(b)) > 0\}$$

is countable. Since the set $H_0[X] \setminus H_0$ is countable, there is an element

$$b \in G \setminus \left(H_0 \cup \bigcup_{w \in H_0[X] \setminus H_0} P_w \right).$$

This element b will remain outside our subgroups throughout the construction.

We proceed by induction. For a limit ordinal α , we set $H_\alpha := \bigcup_{\beta < \alpha} H_\beta$. For a successor ordinal $\alpha = \beta + 1 < \mathfrak{c}$, we assume, inductively, that $|H_\beta| < \mathfrak{c}$, $b \notin H_\beta$, and the sets $w^{-1}(b)$ are null for all words $w \in H_\beta[X] \setminus H_\beta$.

Since $|H_\beta[X]| < \mathfrak{c}$, the set

$$S := \bigcup_{\substack{w \in H_\beta[X] \\ |w|=1}} w^{-1}(b) \cup \bigcup_{\substack{w \in H_\beta[X] \\ |w| \geq 2}} \{g \in G : \mu((w^{-1}(b))_g) > 0\}.$$

is a union of fewer than $\mathfrak{fm}(G)$ Fubini–Markov sets, and thus does not have full measure. Pick an element $g_\alpha \in G \setminus (S \cup N_\alpha)$. Let $H_\alpha := \langle H_\beta, g_\alpha \rangle$. We verify that the inductive hypotheses are preserved.

Fix $c \in H_\alpha$. There is a word $w \in H_\beta[X]$ with $|w| = 1$ such that $w(g_\alpha) = c$, and thus $g_\alpha \in w^{-1}(c)$. Since $g_\alpha \notin S \supseteq w^{-1}(b)$, we have $c \neq b$. This shows that $b \notin H_\alpha$. Next, consider an arbitrary word $v = v(x_1, \dots, x_n) \in H_\alpha[X] \setminus H_\alpha$. There is a word $w = w(x_1, \dots, x_n, x_{n+1}) \in H_\beta[X] \setminus H_\beta$ such that

$$v(x_1, \dots, x_n) = w(x_1, \dots, x_n, g_\alpha).$$

Since $g_\alpha \notin S$ and $|w| \geq 2$, the set $v^{-1}(b) = (w^{-1}(b))_{g_\alpha}$ is null.

Having defined all subgroups H_α for $\alpha < \mathfrak{c}$, let $H := \bigcup_{\alpha < \mathfrak{c}} H_\alpha$. Then H is a proper (since $b \notin H$), dense (since $H_0 \subseteq H$), nonnull (since $H \not\subseteq N_\alpha$ for all α) subgroup of G . Assume that H is measurable. Then it has positive measure, and by the Steinhaus Theorem, it contains an open set. Since it is dense, we have $H = G$, a contradiction. \square

Our main theorem also has a dual, Baire category version. Let \mathcal{M} be the ideal of meager (Baire first category) subsets of the Cantor space. We define *Kuratowski–Ulam–Markov sets* by changing *null* to *meager* in Definition 3. In this case, item (2) of the definition becomes

$$N = \{g \in G : A_g \text{ is nonmeager}\}.$$

By the Kuratowski–Ulam Theorem, Kuratowski–Ulam–Markov sets are meager. Similarly, we dualize Definition 5 to define the Kuratowski–Ulam–Markov number $\mathfrak{kum}(G)$. Let $\text{non}(\mathcal{M})$ be the minimal cardinality of a nonmeager subset of the Cantor space.

Theorem 8. *Let G be an infinite metrizable profinite group with $\text{non}(\mathcal{M}) \leq \mathfrak{kum}(G)$. Then G has a subgroup that does not have the property of Baire.*

Proof. If $\text{non}(\mathcal{M}) < \mathfrak{c}$, then any nonmeager set of cardinality $\text{non}(\mathcal{M})$ generates a nonmeager subgroup of G that does not have the Baire property (nonmeager sets with the Baire property have cardinality \mathfrak{c}) [3, Theorem 2.5].

Thus, assume that $\text{non}(\mathcal{M}) = \mathfrak{c}$. We proceed as in the proof of Theorem 7, replacing *null* by *meager* and sets of positive measure by *nonmeager sets* (the relevant sets are closed). For the choice of the element b , we observe that closed nonmeager sets are, in particular, not nowhere dense, and thus have nonempty interior. Our group G is homeomorphic to the Cantor space, and thus there are in G at most countably many disjoint open sets.

We thus obtain a proper dense nonmeager subgroup of G . To conclude the proof, we use the Pettis Theorem [6, Theorem 9.9], the category-theoretic dual of the Steinhaus Theorem: If a set $H \subseteq G$ is nonmeager and has the Baire property, then the quotient $H^{-1}H$ has nonempty interior. \square

3. BOUNDS ON THE FUBINI–MARKOV NUMBER

Here too, all groups are assumed to be infinite metrizable profinite. Theorem 7 applies to groups G with $\text{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$. We saw that abelian groups have $\mathfrak{fm}(G) = \mathfrak{c}$, but the following conjecture remains open.

Conjecture 9. *For each infinite metrizable profinite group G , we have $\text{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$.*

A proof of this conjecture would settle the Haar Measure Problem, but it may turn out unprovable (and thus undecidable). In this case, well-studied lower bounds on $\mathfrak{fm}(G)$ are useful.

The *covering number* of an ideal \mathcal{I} of subsets of the Cantor space, denoted $\text{cov}(\mathcal{I})$, is the minimal number of elements of \mathcal{I} needed to cover the Cantor space. Since Fubini–Markov sets are null, we have $\text{cov}(\mathcal{N}) \leq \mathfrak{fm}(G)$ for all groups G : A set of full measure needs just one

additional null set to cover the entire space. The following result provides a tighter estimate, in the sense that it is provably larger, and consistently strictly larger.

Let \mathcal{E} be the σ -ideal generated by the closed null sets in the Cantor space. The following proof establishes, in particular, that the family of Fubini–Markov sets is contained in \mathcal{E} .

Proposition 10. *For each infinite metrizable profinite group G , we have $\text{cov}(\mathcal{E}) \leq \mathfrak{fm}(G)$.*

Proof. Brian proved that $\text{cov}(\mathcal{E})$ is equal to the minimal number of closed null subsets of the Cantor space that cover a set of positive measure [2]. Thus, it suffices to prove that every Fubini–Markov subset N of G is a countable union of closed null subsets of G .

Let N be a Fubini–Markov subset of G . If N is Markov null, then it is closed and null. It remains to consider the case that

$$N = \{ g \in G : \mu((w^{-1}(e))_g) > 0 \},$$

where $w \in G[X]$ has $|w| \geq 2$.

For each natural number k , the subset

$$N_k := \{ g \in G : \mu((w^{-1}(e))_g) \geq 1/k \}$$

of N is null (Lemma 4), and $N = \bigcup_k N_k$. Each set N_k is closed: Let $g \in G \setminus N_k$. There is an open set V in $G^{|w|-1}$ such that $(w^{-1}(e))_g \subseteq V$ and $\mu(V) < 1/k$. Let P be the projection of the compact set $w^{-1}(e) \setminus (G^{|w|-1} \times V)$ on the last coordinate. The set $G \setminus P$ is an open neighborhood of g in G . For each element $h \in G \setminus P$, we have $(w^{-1}(e))_h \subseteq V$. Thus, $\mu((w^{-1}(e))_h) \leq \mu(V) < 1/k$, and $(G \setminus P) \cap N_k = \emptyset$. \square

Corollary 11. *Assume that $\text{non}(\mathcal{N}) \leq \text{cov}(\mathcal{E})$. Then every infinite compact group has a nonmeasurable subgroup.*

Proof. Theorem 7 and Proposition 10. \square

Since $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$, we have

$$\max\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\} \leq \text{cov}(\mathcal{E}).$$

It follows that if $\text{non}(\mathcal{N}) \leq \max\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\}$, then every compact group has a nonmeasurable subgroup. The hypothesis $\text{non}(\mathcal{N}) \leq \text{cov}(\mathcal{E})$ is not provable; this follows from known upper bounds on $\text{cov}(\mathcal{E})$ [7].

We conclude this section with a simple sufficient condition for our main theorem. This condition is stronger than the hypothesis $\text{non}(\mathcal{N}) \leq \mathfrak{fm}(G)$, but it may still be provable.

Definition 12. Let G be an infinite metrizable profinite group. For a natural number n , let κ_n be the minimal number of Markov null subsets of the group G^n whose union is not null. The *Markov number* of G is the cardinal number $\mathbf{mar}(G) := \min_n \kappa_n$.

Problem 13. In Definition 12, is the sequence $\kappa_1, \kappa_2, \dots$ constant? In particular, is it provable that $\mathbf{mar}(G)$ is equal to the minimal number of Markov null subsets of the group G whose union is not null?

Lemma 14. Let G be an infinite metrizable profinite group. Then:

$$(1) \text{cov}(\mathcal{M}) \leq \mathbf{mar}(G) \leq \text{non}(\mathcal{N}),$$

$$(2) \mathbf{mar}(G) \leq \mathbf{fm}(G).$$

Proof. (1) Markov sets are closed and null. It follows that the minimal number of closed null sets in the Cantor space whose union is not null is at most $\mathbf{mar}(G)$. The former number is equal to $\text{cov}(\mathcal{M})$ [1, Theorem 2.6.14]. Since every singleton is a Markov set (consider the word $w(x) = x$), we have $\mathbf{mar}(G) \leq \text{non}(\mathcal{N})$.

(2) Let \mathcal{F} be a family of Fubini–Markov subsets of the group G with $|\mathcal{F}| < \mathbf{mar}(G)$. For each element of \mathcal{F} , fix a Markov null set witnessing its being Fubini–Markov, and let \mathcal{A} be the family of these Markov sets. For a natural number n , let $A_n := \bigcup \{ A \in \mathcal{A} : A \subseteq G^n \}$. Since $|\mathcal{A}| < \mathbf{mar}(G)$, the set A_n is null. By the Fubini Theorem, the set

$$S := A_1 \cup \bigcup_{n=2}^{\infty} \{ g \in G : (A_n)_g \text{ is not null} \}$$

is null. Then S is null, and $\bigcup \mathcal{F} \subseteq S$. □

Thus, the following conjecture implies a positive solution to the Haar Measure Problem.

Conjecture 15. For each infinite metrizable profinite group G , we have $\text{non}(\mathcal{N}) = \mathbf{mar}(G)$.

Conjecture 15 holds when restricted to abelian groups, since Lipschitz images of sets of the form $A \times \{0\}$, with $|A| < \text{non}(\mathcal{N})$, are null (see Example 6)

Acknowledgments. We are indebted to Will Brian for answering a question of ours [2]. His answer simplified Proposition 10. We thank the editor and the referee for the evaluation of this paper.

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