Iterated ultrapowers as countably compact groups

Michael Hrušák
(joint with Jan van Mill, Ariet Ramos and Saharon Shelah)

CCM
Universidad Nacional Autónoma de México
michael@matmor.unam.mx

Eilat
2018

Versions of compactness

Given an ultrafilter p on ω, a sequence $\{x_n : n \in \omega\}$ and a point x contained in a topological space X we say that $x = p\text{-}\lim x_n$ if $\{n \in \omega : x_n \in U\} \in p$ for every neighbourhood $U \subseteq X$ of x.

Definition

Let X be a topological space and let p be a free ultrafilter p on ω.

- X is compact if very open cover of X has a finite subcover.
- X is p-compact if for every sequence $\{x_n : n \in \omega\} \subseteq X$ there is a point $x \in X$ such that $x = p\text{-}\lim x_n$.
- X is countably compact if every countable open cover of X has a finite subcover (iff every infinite set has an accumulation point),
- X is pseudo-compact if every continuous function $f : X \to \mathbb{R}$ is bounded.

Versions of compactness and products

- (Tychonoff '30/'35) Any product of compact spaces is compact.
- (Ginsburg-Saks '75) Any product of p-compact spaces is compact.
- (Teresaka '52, Novák '53) There are countably compact spaces whose square is not even pseudo-compact.
- (Comfort-Ross '66) Any product of pseudo-compact topological groups is pseudocompact.

Problem (Comfort '66)

Are there countably compact groups $\mathbb{G}, \mathbb{H}$ such that $\mathbb{G} \times \mathbb{H}$ is not countably compact?

Versions of compactness and products

- (Tychonoff '30/'35) Any product of compact spaces is compact.
- (Ginsburg-Saks '75) Any product of p-compact spaces is compact.
- (Teresaka '52, Novák '53) There are countably compact spaces whose square is not even pseudo-compact.
- (Comfort-Ross '66) Any product of pseudo-compact topological groups is pseudocompact.

Problem (Comfort '66)

Are there countably compact groups $\mathbb{G}, \mathbb{H}$ such that $\mathbb{G} \times \mathbb{H}$ is not countably compact?

Comfort's problem - consistent solutions

Problem (Comfort '66)

Are there countably compact groups $\mathbb{G}, \mathbb{H}$ such that $\mathbb{G} \times \mathbb{H}$ is not countably compact?

- (van Douwen '80) (MA) There are two countably compact subgroups of $2^{\mathfrak{c}}$ whose product is not countably compact.
- (van Mill-Hart '91) (MA_{ctble}) There is a countably compact group whose square is not countably compact.
- (Tomita '99) (MA_{ctble}) There is a group whose square is countably compact but the cube is not.
- (van Douwen '80) Every boolean countably compact group without (non-trivial) convergent sequences contains two countably compact subgroups whose product is not countably compact.
- (Hajnal-Juhász '76) (CH) There is a boolean countably compact group without convergent sequences.

Comfort's problem - consistent solutions

Problem (Comfort '66)

Are there countably compact groups $\mathbb{G}, \mathbb{H}$ such that $\mathbb{G} \times \mathbb{H}$ is not countably compact?

- (van Douwen '80) (MA) There are two countably compact subgroups of $2^{\mathfrak{c}}$ whose product is not countably compact.
- (van Mill-Hart '91) (MA_{ctble}) There is a countably compact group whose square is not countably compact.
- (Tomita '99) (MA_{ctble}) There is a group whose square is countably compact but the cube is not.

- (van Douwen '80) Every boolean countably compact group without (non-trivial) convergent sequences contains two countably compact subgroups whose product is not countably compact.
- (Hajnal-Juhász '76) (CH) There is a boolean countably compact group without convergent sequences.

van Douwen's problem - consistent solutions

Problem (van Douwen '80)

Is there countably compact group without non-trivial convergent sequences?

- (Kuz'minov '58) Every compact topological group contains a convergent sequence.
- (Hajnal-Juhász '76) Yes assuming CH.
- (van Douwen '80) Yes assuming MA.
- (Tomita) Yes assuming MA_{ctble}.
- ...
- (Garcia-Ferreira, Tomita, Watson '05) Yes assuming the existence of a selective ultrafilter, in fact, then there is a p-compact group.
- (Szeptycki, Tomita '09) Yes in the random real model.

All of these constructions describe subgroups of $2^{\mathfrak{c}}$.

Main result

Theorem

There is a countably compact boolean group (subgroup of $2^{\mathfrak{c}}$) without convergent sequences (in ZFC).

Ultrapowers of topological spaces

Given a (zero dimensional) topological space (X, τ) and an ultrafilter $p \in \omega^*$ let

$$ult_p(X) = (X^\omega / \equiv_p) \text{ where } f \equiv_p g \text{ iff } \{n : f(n) = g(n)\} \in p.$$

endowed it with the topology which has as a basis the collection $\{U^* : U \in \mathcal{B}\}$, where

$$U^* = \{[f] \in ult_p(X) : \{n : f(n) \in U\} \in p\}$$

for some basis $\mathcal{B}$ of τ ($\mathcal{B} \subseteq Clop(\tau)$). The ultrapower with this topology is usually not Hausdorff, so we identify the inseparable functions and denote by $Ult_p(X)$ this quotient.

Note that X is a dense subset of $Ult(X)$ and every sequence in X has a p-limit in $Ult_p(X)$ ($p\text{-lim}\, f(n) = [f]$).

Iterated ultrapowers

The process can, of course, be iterated:

Given a space X, ultrafilter $p \in \omega^*$ and $\alpha < \omega_1$ let

$$Ult_p^\alpha(X) = Ult_p\Big(\bigcup_{\beta<\alpha} Ult_p^\beta(X)\Big)$$

and finally

$$Ult_p^{\omega_1}(X) = \bigcup_{\beta<\omega_1} Ult_p^\beta(X).$$

Note that

- $Ult_p^\alpha(X) = Ult_{p^\alpha}(X)$ for $\alpha < \omega_1$,
- $Ult_p^{\omega_1}(X)$ is p-compact, and
- if X is p-compact then $X = Ult_p^{\omega_1}(X)$ (in particular, all iterations beyond ω_1 do not produce new spaces).

The plan

Theorem

There is a countably compact boolean group (subgroup of $2^{\mathfrak{c}}$) without convergent sequences (in ZFC).

Fix an ultrafilter $p \in \omega^*$, find a suitable topological group $\mathbb{G}$ without convergent sequences and consider $Ult_p^{\omega_1}(\mathbb{G})$.

Then $Ult_p^{\omega_1}(\mathbb{G})$ is a p-compact space and a group.

There are two problems:

- Is $Ult_p^{\omega_1}(\mathbb{G})$ with the ultraproduct topology a topological group?
- Does $Ult_p^{\omega_1}(\mathbb{G})$ have convergent sequences?

The plan

Theorem

There is a countably compact boolean group (subgroup of $2^{\mathfrak{c}}$) without convergent sequences (in ZFC).

Fix an ultrafilter $p \in \omega^*$, find a suitable topological group $\mathbb{G}$ without convergent sequences and consider $Ult_p^{\omega_1}(\mathbb{G})$.

Then $Ult_p^{\omega_1}(\mathbb{G})$ is a p-compact space and a group.

There are two problems:

- Is $Ult_p^{\omega_1}(\mathbb{G})$ with the ultraproduct topology a topological group?
- Does $Ult_p^{\omega_1}(\mathbb{G})$ have convergent sequences?

Group topology on $[\omega]^{<\omega}$

Let

$$Hom([\omega]^{<\omega}) = \{\Phi : \ \Phi \text{ is a group homomorphism from } [\omega]^{<\omega} \text{ to } 2\}$$

and let τ_{Hom} be the topology induced by $Hom([\omega]^{<\omega})$.

- $([\omega]^{<\omega}, \tau_{Hom})$ is homeomorphic to a countable dense subgroup of $2^{\mathfrak{c}}$ without convergent sequences.
- $Ult_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact topological group for every p in ω^* (see next page).

Extensions of homomorphisms to ltrapowers

Every $\Phi \in Hom([\omega]^{<\omega})$ naturally extends to a homomorphism $\overline{\Phi} \in Hom(Ult_p([\omega]^{<\omega}))$ by letting

$$\overline{\Phi}([f]_p) = i \text{ iff } \{k : \Phi(f(k)) = i\} \in p.$$

The ultrapower topology is the topology induced by $\{\bar{\Phi} : \Phi \in Hom([\omega]^{<\omega})\}$ on $Ult_p([\omega]^{<\omega})$.

Similarly, the utrapower topology on $Ult_p^{\omega_1}([\omega]^{<\omega})$ is the topology $\tau_{\overline{Hom}}$ induced by the homomorphisms in $Hom([\omega]^{<\omega})$ extended recursively all the way to $Ult_p^{\omega_1}([\omega]^{<\omega})$ by the same formula as before:

$$\overline{\Phi}([f]) = i \text{ if and only if } \{k : \overline{\Phi}(f(k)) = i\} \in p.$$

Fact.

$Ult_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact topological group.

The plan works ... for selective ultrafilters

Proposition

p is selective iff for every $\{f_n : n \in \omega\}$ of functions $f_n : \omega \to [\omega]^{<\omega}$ which are not constant or equal on an element of p, there is a sequence $\{U_n : n \in \omega\} \subseteq p$ such that

$$\{f_n(m) : n \in \omega \text{ and } m \in U_n\}$$

is linearly independent.

Corollary

If p is selective then $Ult_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact topological group without convergent sequences.

Different, yet the same

Lemma

There is a family $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^*$ such that for every $D \in [\mathfrak{c}]^\omega$ and every sequence $\{f_\alpha : \alpha \in D\} \subseteq ([\omega]^{<\omega})^\omega$ of one-to-one enumerations of linearly independent sets there are $\{U_\alpha : \alpha \in D\}$ such that

1. $\forall \alpha \in D \; U_\alpha \in p_\alpha$, and
2. $\{f_\alpha(n) : \alpha \in D \;\&\; n \in U_\alpha\}$ is a linearly independent subset of $[\omega]^{<\omega}$.

Construct a countably compact topology on $[\mathfrak{c}]^{<\omega}$ starting from $([\omega]^{<\omega}, \tau_{Hom})$ as follows: Fix a family $\{f_\alpha : \omega \leq \alpha < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ s, t.

1. for every infinite $X \subseteq [\mathfrak{c}]^{<\omega}$ there is an $\alpha < \omega_1$ with $rng(f_\alpha) \subseteq X$,
2. Each f_α is a one-to-one enumeration of a linearly independent set, and
3. for every $\alpha < \omega_1$ $rng(f_\alpha) \subseteq [\alpha]^{<\omega}$.

For every $\Phi \in Hom([\omega]^{<\omega})$ define its extension $\overline{\Phi} \in Hom([\mathfrak{c}]^{<\omega})$ recursively by putting

$$\overline{\Phi}(\{\alpha\}) = p_\alpha\text{-}\lim \overline{\Phi}(f_\alpha(n)).$$

with the topology $\tau_{\overline{Hom}}$ induced by $\{\bar{\Phi} : \Phi \in Hom([\omega]^{<\omega})\}$ on $[\mathfrak{c}]^{<\omega}$.

Different, yet the same

Lemma

There is a family $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^*$ such that for every $D \in [\mathfrak{c}]^\omega$ and every sequence $\{f_\alpha : \alpha \in D\} \subseteq ([\omega]^{<\omega})^\omega$ of one-to-one enumerations of linearly independent sets there are $\{U_\alpha : \alpha \in D\}$ such that

1. $\forall \alpha \in D \;\; U_\alpha \in p_\alpha$, and
2. $\{f_\alpha(n) : \alpha \in D \;\&\; n \in U_\alpha\}$ is a linearly independent subset of $[\omega]^{<\omega}$.

Construct a countably compact topology on $[\mathfrak{c}]^{<\omega}$ starting from $([\omega]^{<\omega}, \tau_{Hom})$ as follows: Fix a family $\{f_\alpha : \omega \leq \alpha < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ s, t.

1. for every infinite $X \subseteq [\mathfrak{c}]^{<\omega}$ there is an $\alpha < \omega_1$ with $rng(f_\alpha) \subseteq X$,
2. Each f_α is a one-to-one enumeration of a linearly independent set, and
3. for every $\alpha < \omega_1$ $rng(f_\alpha) \subseteq [\alpha]^{<\omega}$.

For every $\Phi \in Hom([\omega]^{<\omega})$ define its extension $\overline{\Phi} \in Hom([\mathfrak{c}]^{<\omega})$ recursively by putting

$$\overline{\Phi}(\{\alpha\}) = p_\alpha\text{-}\lim \overline{\Phi}(f_\alpha(n)).$$

with the topology $\tau_{\overline{Hom}}$ induced by $\{\overline{\Phi} : \Phi \in Hom([\omega]^{<\omega})\}$ on $[\mathfrak{c}]^{<\omega}$.

Different, yet the same

Lemma

There is a family $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^*$ such that for every $D \in [\mathfrak{c}]^\omega$ and every sequence $\{f_\alpha : \alpha \in D\} \subseteq ([\omega]^{<\omega})^\omega$ of one-to-one enumerations of linearly independent sets there are $\{U_\alpha : \alpha \in D\}$ such that

1. $\forall \alpha \in D\ U_\alpha \in p_\alpha$, and
2. $\{f_\alpha(n) : \alpha \in D \ \&\ n \in U_\alpha\}$ is a linearly independent subset of $[\omega]^{<\omega}$.

Construct a countably compact topology on $[\mathfrak{c}]^{<\omega}$ starting from $([\omega]^{<\omega}, \tau_{Hom})$ as follows: Fix a family $\{f_\alpha : \omega \leq \alpha < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ s, t.

1. for every infinite $X \subseteq [\mathfrak{c}]^{<\omega}$ there is an $\alpha < \omega_1$ with $rng(f_\alpha) \subseteq X$,
2. Each f_α is a one-to-one enumeration of a linearly independent set, and
3. for every $\alpha < \omega_1$ $rng(f_\alpha) \subseteq [\alpha]^{<\omega}$.

For every $\Phi \in Hom([\omega]^{<\omega})$ define its extension $\overline{\Phi} \in Hom([\mathfrak{c}]^{<\omega})$ recursively by putting

$$\overline{\Phi}(\{\alpha\}) = p_\alpha\text{-}\lim \overline{\Phi}(f_\alpha(n)).$$

with the topology $\tau_{\overline{Hom}}$ induced by $\{\bar{\Phi} : \Phi \in Hom([\omega]^{<\omega})\}$ on $[\mathfrak{c}]^{<\omega}$.

Different, yet the same

Call a set $D \in [\mathfrak{c}]^\omega$ *suitably closed* if $\omega \subseteq D$ and $\bigcup_{n\in\omega} f_\alpha(n) \subseteq D$ for every $\alpha \in D$.

Proposition

The topology $\tau_{\overline{Hom}}$ contains no non-trivial convergent sequences if and only if
$\forall D \in [\mathfrak{c}]^\omega$ *suitably closed* $\forall X \in [D]^\omega\ \exists \Psi \in Hom([D]^{<\omega})$ *such that*

1. $\forall \alpha \in D\ \Psi(\{\alpha\}) = p_\alpha\text{-}\lim \Psi(f_\alpha(n))$
2. $|X \cap Ker(\Psi)| = |X \setminus Ker(\Psi)| = \omega$.

Now, if this happens (and it does by our choice of the ultrafilters) then, in particular,

$$K = \bigcap_{\Phi \in Hom([\omega]^{<\omega})} Ker(\overline{\Phi})$$

is finite, and $[\mathfrak{c}]^{<\omega}/K$ with the quotient topology is the Hausdorff countably compact group without non-trivial convergent sequences we want.

Final remarks and questions

Theorem

For every $n \in \omega$ there is a group $\mathbb{G}$ such that $\mathbb{G}^n$ is countably compact while $\mathbb{G}^{n+1}$ is not.

Question

(1) Is there a countably compact group $\mathbb{G}$ without convergent sequences which is not a torsion group, i.e contains a copy of $\mathbb{Z}$?

(2) (Wallace '55) Is every countably compact semigroup with both-sided cancellation a topological group?

Yes to (1) $\Rightarrow$ Yes to (2).

Question

Is there consistently a countably compact group $\mathbb{G}$ without convergent sequences of weight less than $\mathfrak{c}$?

Question

Is there a p-compact group $\mathbb{G}$ without convergent sequences in ZFC?

Final remarks and questions

Theorem

For every $n \in \omega$ there is a group $\mathbb{G}$ such that $\mathbb{G}^n$ is countably compact while $\mathbb{G}^{n+1}$ is not.

Question

(1) Is there a countably compact group $\mathbb{G}$ without convergent sequences which is not a torsion group, i.e contains a copy of $\mathbb{Z}$?

(2) (Wallace '55) Is every countably compact semigroup with both-sided cancellation a topological group?

Yes to (1) $\Rightarrow$ Yes to (2).

Question

Is there consistently a countably compact group $\mathbb{G}$ without convergent sequences of weight less than $\mathfrak{c}$?

Question

Is there a p-compact group $\mathbb{G}$ without convergent sequences in ZFC?

Final remarks and questions

Theorem

For every $n \in \omega$ there is a group $\mathbb{G}$ such that $\mathbb{G}^n$ is countably compact while $\mathbb{G}^{n+1}$ is not.

Question

(1) Is there a countably compact group $\mathbb{G}$ without convergent sequences which is not a torsion group, i.e contains a copy of $\mathbb{Z}$?

(2) (Wallace '55) Is every countably compact semigroup with both-sided cancellation a topological group?

Yes to (1) $\Rightarrow$ Yes to (2).

Question

Is there consistently a countably compact group $\mathbb{G}$ without convergent sequences of weight less than $\mathfrak{c}$?

Question

Is there a p-compact group $\mathbb{G}$ without convergent sequences in ZFC?

Final remarks and questions

Theorem

For every $n \in \omega$ there is a group $\mathbb{G}$ such that $\mathbb{G}^n$ is countably compact while $\mathbb{G}^{n+1}$ is not.

Question

(1) Is there a countably compact group $\mathbb{G}$ without convergent sequences which is not a torsion group, i.e contains a copy of $\mathbb{Z}$?

(2) (Wallace '55) Is every countably compact semigroup with both-sided cancellation a topological group?

Yes to (1) $\Rightarrow$ Yes to (2).

Question

Is there consistently a countably compact group $\mathbb{G}$ without convergent sequences of weight less than $\mathfrak{c}$?

Question

Is there a p-compact group $\mathbb{G}$ without convergent sequences in ZFC?

Thank you for your attention!