On some topological concepts related with tightness

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Theorem 1 (Hrusak-Ramos Garcia (Malykhin problem))

There exists a model in ZFC where every separable Fréchet-Urysohn group is metrizable.

4. This motivates the following problem:

Problem 2

Describe possible good sufficient conditions under which every Fréchet-Urysohn group is metrizable.

1. In general topology there are some concepts strictly located between metrizability and countably tightness which together with the Fréchet-Urysohn property imply metrizability for topological groups.

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2. Strong Pytkeev property (or even countable cs^*-character) is one of possible such concepts.
3. Will be shown how to use the concept of a $\mathfrak{G}$-base while studying (Strong)Pytkeev property.

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Definition 3 (Malyhin-Tironi, Pytkeev)

A topological space X has the **Pytkeev property** if for each $A \subseteq X$ and each $x \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \ldots$ of A such that each neighbourhood of x contains some A_n.

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Definition 4 (Tsaban–Zdomskyy)

A topological space X has the **strong Pytkeev property** if for each $x \in X$, there exists a countable family $\mathcal{D}$ of subsets of X such that for each neighbourhood U of x and each $A \subseteq X$ with $x \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite.

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Theorem 6 (Sakai)

For Tychonoff X the following are equivalent:

(i) $C_p(X)$ *has the strong Pytkeev property.*

(ii) $C_p(X)$ *has countable cs^*-network.*

(iii) X *is countable.*

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Definition 7 (Cascales–Kąkol–Saxon)

G – topological group. A family $\mathcal{U} := (U_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a **$\mathfrak{G}$-base** if $\mathcal{U}$ is a base of neighb. at e and $U_\beta \subseteq U_\alpha$ for $\alpha \leq \beta$.

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Theorem 8 (Gabriyelyan–Kąkol-Leiderman)

G is metrizable iff G is Fréchet-Urysohn with a $\mathfrak{G}$-base. Any precompact set in G with a $\mathfrak{G}$-base is metrizable.

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Theorem 9 (Ferrando–Kąkol)

$C_c(X)$ has a $\mathfrak{G}$-base iff X has a compact resolution $(K_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ swallowing compact sets of X, i.e. $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$.

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Theorem 10 (Cascales–Kąkol–Saxon)

$C_p(X)$ has a $\mathfrak{G}$-base iff X is countable.

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(vii) X *is a μ-space.*

Corollary 12

Let X be a Čech-complete space. Then $C_c(X)$ has the strong Pytkeev property if and only if X is Lindelöf.

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E with a $\mathfrak{G}$-base $\Rightarrow$ E is a k-space iff E is metrizable or $E \approx \phi$ or $E \approx \phi \times Q$, where Q- Hilbert cube (**G-Ką-L**).

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Example 14 (Banakh)

There exists a topological group G with the strong Pytkeev property but without a $\mathfrak{G}$-base.

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(iii) $U_\beta \subseteq U_\alpha$, *whenever* $\alpha \leq \beta$ *for* $\alpha, \beta \in \mathsf{M}$.

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2. $C_c(\mathbb{Q})$ has a base $\{U_\alpha : \alpha \in \mathbf{M}\}$ over $\mathbf{M}$ such that $\sup\{\alpha(K) : \alpha \in \mathbf{M}\} = \infty$ for any $k \in \mathbb{N}$ but $C_c(\mathbb{Q})$ does not admit any $\mathfrak{G}$-base (**Ferrando–Kąkol**).

Theorem 18 (Gabriyelyan–Kąkol–Leiderman)

(i) *A (DF)-space E has countable tightness iff E has the strong Pytkeev property.*

(ii) *Every strict (LM)-space has the strong Pytkeev property.*

(iii) *The strong dual E'_β of a strict (LM)-space has the strong Pytkeev property iff E'_β has countable tightness,*

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2. $\mathfrak{D}'(\Omega)$ has the strong Pytkeev property, since it has countable tightness.
3. If $E := C_c(X)$ is metrizable and complete, then E'_β has the strong Pytkeev property (since has countable tightness).

1. For $\alpha = (\alpha_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $k \in \mathbb{N}$, set
   $I_k(\alpha) := \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta_i = \alpha_i,\ i = 1, \ldots, k\}$.

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2. Let $\mathbf{M} \subseteq \mathbb{N}^{\mathbb{N}}$ and $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ be an $\mathbf{M}$-decreasing family of subsets of a set Ω. Define the (countable) family $\mathcal{D}_{\mathcal{U}}$ of subsets of Ω by

$$\mathcal{D}_{\mathcal{U}} := \{D_k(\alpha) : \alpha \in \mathbf{M}, k \in \mathbb{N}\},$$

where $D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}} U_\beta$, and say that $\mathcal{U}$ satisfies the *condition* (**D**) if $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ for every $\alpha \in \mathbf{M}$.

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where $D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}} U_\beta$, and say that $\mathcal{U}$ satisfies the *condition* $(\mathbf{D})$ if $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ for every $\alpha \in \mathbf{M}$.
3. A lsc with countable tightness with a $\mathfrak{G}$-base admits $\mathcal{U}$ with $(\mathbf{D})$ and $\mathbf{M} = \mathbb{N}^{\mathbb{N}}$. Every (DF)-space with countable tightness has this property (**Cascales–Kąkol–Saxon**).

A global concept.

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A topological space X has a **small base** if there exists an $\mathbf{M}$-decreasing base of τ for some $\mathbf{M} \subseteq \mathbb{N}^{\mathbb{N}}$.

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(i) *X is cosmic iff X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ with condition $(\mathbf{D})$. The family $\mathcal{D}_{\mathcal{U}}$ is a countable network in X.*

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Definition 19

A topological space X has a **small base** if there exists an $\mathbf{M}$-decreasing base of τ for some $\mathbf{M} \subseteq \mathbb{N}^{\mathbb{N}}$.

Theorem 20 (Gabriyelyan–Kąkol)

Let $X := (X, \tau)$ be a regular topological space. Then:

(i) *X is cosmic iff X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ with condition $(\mathbf{D})$. The family $\mathcal{D}_{\mathcal{U}}$ is a countable network in X.*

(ii) *X is an $\aleph_0$-space iff X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ with condition $(\mathbf{D})$ such that $\mathcal{D}_{\mathcal{U}}$ is a countable k-network in X.*

Corollary 21

A Baire separable group G is metrizable iff G is cosmic.

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Theorem 22 (Gabriyelyan–Kąkol)

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Theorem 22 (Gabriyelyan–Kąkol)

Let G be a Baire group. The following are equivalent:

(i) *G is metrizable.*

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A Baire separable group G is metrizable iff G is cosmic.

Theorem 22 (Gabriyelyan–Kąkol)

Let G be a Baire group. The following are equivalent:

(i) *G is metrizable.*

(ii) *G has the strong Pytkeev property.*

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A Baire separable group G is metrizable iff G is cosmic.

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Let G be a Baire group. The following are equivalent:

(i) *G is metrizable.*

(ii) *G has the strong Pytkeev property.*

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(i) *G is metrizable.*

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(iv) *G has countable tightness and has a $\mathfrak{G}$-base*

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The equivalence "(iii) iff (iv)" holds for any topological group. **[Banakh]**

Theorem 23 (Banakh–Leiderman)

A locally precompact group is metrizable iff it contains a dense subgroup with the strong Pytkeev property.

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Theorem 24 (Gabriyelyan–Kąkol–Leiderman)

A locally precompact group G is metrizable iff G has a $\mathfrak{G}$-base.

What about Banach spaces E endowed with the weak topology $w := \sigma(E, E')$?

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1. A Banach space E is finite-dimensional iff E_w has the Pytkeev property [**G–Ką–Z**]
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(iv) *E' is separable.*

On the dual of a Banach space E we have three topologies: $\sigma(E', E), \mu(E', E)$, the dual norm topology. All three have the same bounded sets. We add also the following

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Theorem 26 (Schüchtermann-Wheller)

For a Banach space E not containing ℓ_1 the dual unit ball B^* is $\mu(E', E)$ metrizable iff E is reflexive.