The Meta-Structure of Universes

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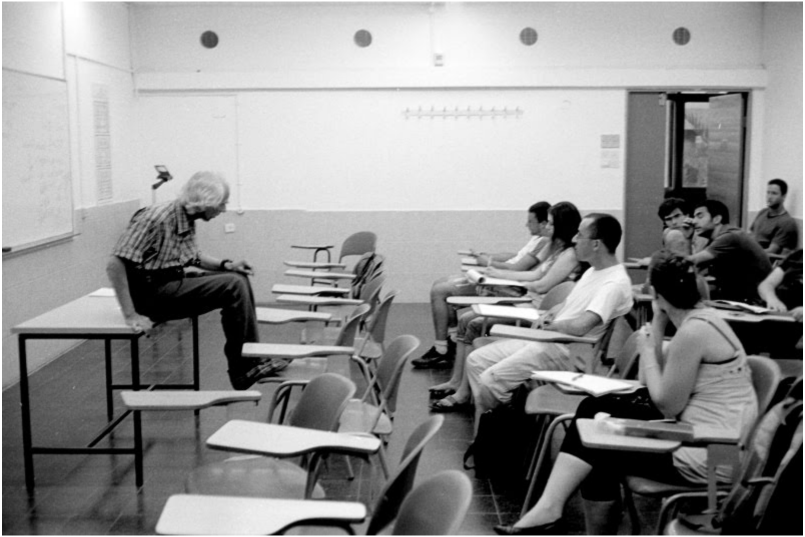
In memoriam of my teacher, mentor, and good friend
Matti Rubin



Matti Rubin, teaching Mathematical Logic, Spring 2009.

Theorem (Intermediate model theorem)

If $V \subseteq M \subseteq V[G]$ are models of ZFC *and $G \subseteq \mathbb{P}$ is a V-generic filter for $\mathbb{P} \in V$, then M is a generic extension of V and $V[G]$ is a generic extension of M.*

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Let's digress for a moment.

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Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ be a symmetric system, then HS denotes the class of hereditarily symmetric names, and if G is a V-generic filter, then $\mathsf{HS}^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}\}$ is a transitive class of $V[G]$ satisfying ZF and $V \subseteq \mathsf{HS}^G$.

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Theorem (Grigorieff)

If $V \subseteq M \subseteq V[G]$ are models of ZF *where $V[G]$ is a generic extension of V and M is a symmetric extension of V, then $V[G]$ is a generic extension of M by a homogeneous forcing.*

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If we assume GCH, that means that there are no more than $\aleph_3$ intermediate models to adding a single Cohen real which are symmetric extensions.

Question

Do we exhaust all the intermediate models of a single Cohen real by considering symmetric extensions?

No.

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The name is due to the fact the model was originally conceived in a workshop in Bristol back in 2011, whose participants were: Andrew Brooke-Taylor, James Cummings, Moti Gitik, Ralf Schindler, Menachem Magidor, Philip Welch, and Hugh Woodin.

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9. Repeat until there are no more ordinals.

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Theorem

If M is a symmetric extension of N, then M satisfies KWP *if and only if N does.*

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This shows that there is a significant difference between symmetric extensions and models of the form $L(x)$, or generally $V(x)$.

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And Woodin's Axiom of Choice Conjecture, if true, would imply that even in case we can construct a Bristol model when an extendible cardinal is present in the universe, it would not retain its largeness inside the Bristol model itself.

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Does the Bristol model work when starting with a different real? Are there other constructions for each step along the way?

Thank you!