Orbits of families of entire functions

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Set Theory, Model Theory and Applications
(In memory of Mati Rubin)
EILAT, April 2018

CH and Entire functions

Definition

A function $f : \mathbb{C} \to \mathbb{C}$ is entire iff it has an everywhere convergent power series expansion:

$$f(z) = \sum_{n<\omega} a_n z^n$$

Question (Wetzel, 1962)

Is there an uncountable family $\mathcal{F}$ of entire functions such that for every $z \in \mathbb{C}$, $\{f(z) : f \in \mathcal{F}\}$ is countable?

Fact

There is a family $\mathcal{F}$ of infinitely differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that $|\mathcal{F}| = \mathfrak{c}$ and for each $x \in \mathbb{R}^n$, $|\{f(x) : f \in \mathcal{F}\}| \leq 2$.

CH and Entire functions

Theorem (Erdős, 1964)

The continuum hypothesis (CH) is equivalent to the existence of an uncountable family $\mathcal{F}$ of entire functions such that $\{f(z) : f \in \mathcal{F}\}$ is countable for every $z \in \mathbb{C}$.

Proof: First suppose CH fails and let $\langle f_i : i < \omega_1 \rangle$ be a list of pairwise distinct entire functions. We'll find a $w \in \mathbb{C}$ such that $\langle f_i(w) : i < \omega_1 \rangle$ has pairwise distinct members. Note that for every $i < j < \omega_1$, the set $A_{i,j} = \{z \in \mathbb{C} : f_i(z) = f_j(z)\}$ is countable (zeros of an entire function cannot accumulate). Choose $w \in \mathbb{C} \setminus \bigcup_{i<j<\omega_1} A_{i,j}$.

CH and Entire functions

Next assume CH. Fix a countable dense set $B \subseteq \mathbb{C}$. It suffices to show the following: For every countable $A \subseteq \mathbb{C}$, there is a non constant entire function f that maps A into B. Let $\langle z_i : i < \omega_1 \rangle$ list $\mathbb{C}$. Construct $\langle f_i : i < \omega_1 \rangle$ such that each f_i is an entire function mapping $\{z_j : j < \alpha_i\}$ into B where $\alpha_i > i$ is large enough to ensure $f_i \notin \{f_j : j < i\}$.
Let $\langle a_n : n < \omega \rangle$ list A. Define

$$f(z) = b_0 + \varepsilon_0(z - a_0) + \varepsilon_1(z - a_0)(z - a_1) + \dots$$

where ε_n's are converging to zero fast enough to guarantee that the series converges uniformly on compact sets. Since B is dense, it is easy to arrange that $f(a_n) \in B$. Note that $f \restriction A$ is an injection from A to B. If A is also dense in $\mathbb{C}$, then we can further ensure that $f \restriction A$ is a bijection from A to B.

When CH fails

Question (Erdős, 1964)

Assume CH fails. Is there a family $\mathcal{F}$ of entire functions such that $|\mathcal{F}| = \mathfrak{c}$ and $|\{f(z) : f \in \mathcal{F}\}| < \mathfrak{c}$ for every $z \in \mathbb{C}$?

Theorem (Kumar-Shelah, 2017)

The existence of a family $\mathcal{F}$ of entire functions such that $|\mathcal{F}| = \mathfrak{c}$ and $(\forall z \in \mathbb{C})\,(|\{f(z) : f \in \mathcal{F}\}| < \mathfrak{c})$ is undecidable in ZFC plus the negation of CH.

Proof ideas I

Claim

There is no such family in the Cohen real model.

Proof: Assume $V \models \mathfrak{c} = \aleph_2$ and let $\mathbb{P}$ be the forcing for adding $\aleph_1$ Cohen reals $\overline{c} = \langle c_i : i < \omega_1 \rangle$ where each $c_i \in \mathbb{C}$. Note that $V[\overline{c}] \models \mathfrak{c} = \aleph_2$. Suppose $\langle f_\alpha : \alpha < \omega_2 \rangle$ is a sequence of pairwise distinct entire functions in $V[\overline{c}]$. Choose $i_\star < \omega_1$ and $X \subseteq \omega_2$ such that for every $\alpha \in X$, f_α is coded in $V[\overline{c} \restriction i_\star]$. It follows that $\langle f_\alpha(c_{i_\star}) : \alpha \in X \rangle$ has pairwise distinct members.

Proof ideas II

Starting with a model where $\mathfrak{c} = \aleph_{\omega_1}$, we perform a finite support iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \omega_1 \rangle$ such that, at each stage $i < \omega_1$, via a ccc forcing $\mathbb{Q}_i$ of size ω_{i+1}, we add a family $\mathcal{F}_i$ of entire functions such that $|\mathcal{F}_i| = \omega_{i+1}$ and for every $j \leq i$, letting W_j be the set of first ω_{j+1} members of $V^{\mathbb{P}_i} \cap \mathbb{C}$ in some fixed enumeration, we have that $(\forall z \in W_j)(|\{f(z) : f \in \mathcal{F}_i\}| \leq \omega_{j+1})$. So $\mathcal{F} = \bigcup\{\mathcal{F}_i : i < \omega_1\}$ will be the required family in $V^{\mathbb{P}}$.

The possible set of values for $\{f(z) : f \in \mathcal{F}_i\}$ is not fixed beforehand but added generically together with $\mathcal{F}$ - This is the major point of difference with Erdős' construction. The main problem then is to ensure that each $\mathbb{Q}_i$ is ccc. We do this by requiring that the finite approximations to members of $\{f(z) : z \in W_i\}$ can be chosen quite independently of those for $\{g(z) : z \in W_i\}$, for $f \neq g \in \mathcal{F}_i$. This is materialized by using strongly almost disjoint families in $[\omega_{i+1}]^{\omega_{i+1}}$.

Proof ideas III

Lemma

The following is consistent.

(a) $\mathfrak{c} = \aleph_{\omega_1}$

(b) *There is a family* $\{A_\alpha : \alpha < \omega_{\omega_1}\}$ *of subsets of* ω_{ω_1} *such that for every* $i < \omega_1$ *and* $\alpha < \omega_{\omega_1}$, $|A_\alpha \cap \omega_{i+1}| = \omega_{i+1}$ *and for every* $\alpha < \beta < \omega_{\omega_1}$, $A_\alpha \cap A_\beta$ *is finite.*

The proof uses Baumgartner's "thinning out forcing".

Proof ideas IV

Lemma

Suppose κ is regular uncountable. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be such that for every $\alpha < \beta < \kappa$ and uncountable cardinal $\mu \leq \kappa$, $A_\alpha \cap \mu \in [\mu]^\mu$ and $A_\alpha \cap A_\beta$ is finite (so $\kappa \leq \mathfrak{c}$). Let $\langle y_\alpha : \alpha < \kappa \rangle$ be a sequence of distinct complex numbers. Then there exists a ccc forcing $\mathbb{Q}$ of size κ such that the following hold in $V^{\mathbb{Q}}$.

(a) *There is a family $\mathcal{F}$ of entire functions of size κ.*

(b) *For every uncountable cardinal $\mu \leq \kappa$,*

$$|\{f(y_\alpha) : \alpha < \mu, f \in \mathcal{F}\}| = \mu$$

A question

Question

Is the following consistent?

(1) $\mathfrak{c} = \aleph_2$ *and*

(2) *there exists* $U \in [\mathbb{C}]^{\aleph_1}$ *such that for every* $X \in [\mathbb{C}]^{\aleph_1}$, *there is a non constant entire function sending* X *into* U.

References

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