Reconstructing structures from their abstract clones

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Set Theory, Model Theory and Applications

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Outline

- Reconstructing structures from their automorphism groups and polymorphism clones
- The topology of algebras
- Reconstruction notions, results, problems

Part I

Reconstructing structures from their automorphism groups and polymorphism clones

countable, ω-categorical

Aut() $\rightarrow$  first-order interdefinable with 

Theorem (Ryll-Nardzewski)

Let Δ, Γ be ω-categorical structures on the same domain.
Then $\mathrm{Aut}(\Delta) = \mathrm{Aut}(\Gamma) \Leftrightarrow \Delta, \Gamma$ are first-order interdefinable.

Aut() as an abstract group $\rightarrow$?

- Can we reconstruct an ω-categorical structure Δ from the algebraic group structure of Aut(Δ)?
- Can we reconstruct the topological structure of Aut(Δ) from its algebraic structure?

Better reconstruction plans

Let Δ be a structure.

- $\mathrm{Aut}(\Delta)\ldots$ automorphism group of Δ
- $\mathrm{End}(\Delta)\ldots$ endomorphism monoid of Δ
- $\mathrm{Pol}(\Delta)\ldots$ polymorphism clone of Δ

$\mathrm{End}(\Delta)$ ... all homomorphisms $f\colon \Delta \to \Delta$.

$\mathrm{Pol}(\Delta)$ ... all homomorphisms $f\colon \Delta^n \to \Delta$, where $1 \leq n < \omega$.

$\mathrm{Pol}(\Delta)$ is a function clone:

- closed under composition
- contains projections.

Observe: $\mathrm{Pol}(\Delta) \supseteq \mathrm{End}(\Delta) \supseteq \mathrm{Aut}(\Delta)$.

Reconstruction up to primitive positive definitions

Pol() $\to$?

Theorem (Bodirsky + Nešetřil '03)

Let Δ, Γ be ω-categorical structures on the same domain.
Then $\text{Pol}(\Delta) = \text{Pol}(\Gamma) \;\Leftrightarrow\; \Delta, \Gamma$ are primitive positive interdefinable.

Why primitive positive definitions?

For Δ a structure with a finite relational signature τ:

Definition (Constraint Satisfaction Problem)

CSP(Δ) is the computational problem to decide whether a given primitive positive τ-sentence holds in Δ.

Topological clones

Function clones carry:

- algebraic structure (composition / equations)
- topological structure (pointwise convergence)

Let **C**, **D** be function clones.

$\xi\colon \mathbf{C} \to \mathbf{D}$ is a (clone) homomorphism iff

- it preserves arities;
- sends every projection in **C** to the corresponding projection in **D**;
- $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$ for all $f, g_1, \ldots, g_n \in \mathbf{C}$.

$\Longrightarrow$ Topological clones

Theorem (Bodirsky + MP '12)

Let Δ, Γ be ω-categorical structures. Then:
$\mathrm{Pol}(\Delta) \cong^T \mathrm{Pol}(\Gamma) \;\Leftrightarrow\; \Delta, \Gamma$ are primitive positive bi-interpretable.

Reconstruction from the abstract clone

Pol() as an abstract clone $\rightarrow$?

- Can we reconstruct an ω-categorical structure Δ from the algebraic clone structure of Pol(Δ)?

- Can we reconstruct the topological structure of Pol(Δ) from its algebraic structure?

Part II

The topology of algebras

Clones from algebras

Let $\mathfrak{A}$ be an algebra.

Term functions of $\mathfrak{A}$ (obtained by composition): function clone $\mathrm{Clo}(\mathfrak{A})$.

$\mathrm{Clo}(\mathfrak{A})$ encodes the equations (=identities) which hold in $\mathfrak{A}$.

Universal Algebra: Structure of $\mathfrak{A} \Leftrightarrow$ equations in $\mathrm{Clo}(\mathfrak{A})$.

Birkhoff's theorem

For an algebra $\mathfrak{A}$ consider the algebras obtained by taking

- Homomorphic images
- Subalgebras
- Powers / finite Powers.

Theorem (Birkhoff 1935)

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras.

Then $\mathrm{Clo}(\mathfrak{B}) = \mathrm{Clo}(\mathfrak{C})$ for some $\mathfrak{C} \in \mathrm{HSP}(\mathfrak{A}) \leftrightarrow$
$\exists$ clone homomorphism from $\mathrm{Clo}(\mathfrak{A})$ onto $\mathrm{Clo}(\mathfrak{B})$.

Theorem (Bodirsky + MP '11)

Let $\mathfrak{A}$, $\mathfrak{B}$ be countable.

Then $\mathrm{Clo}(\mathfrak{B}) = \mathrm{Clo}(\mathfrak{C})$ for some $\mathfrak{C} \in \mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A}) \leftrightarrow$
$\exists$ uniformly continuous clone homomorphism from $\mathrm{Clo}(\mathfrak{A})$ onto $\mathrm{Clo}(\mathfrak{B})$.

HSP vs. HSP^{fin}

- When do HSP and HSP^{fin} coincide for an algebra?
- When can HSP^{fin} be described algebraically?
- Can we reconstruct the topological structure of function clones from their algebraic structure?

Part III

Reconstruction notions & results

Reconstruction notions

Let **O** be the largest function clone on ω, and **C** be a closed subclone.

Definition

- **C** has reconstruction $\Leftrightarrow$ $\mathbf{C} \cong \mathbf{D}$ implies $\mathbf{C} \cong^T \mathbf{D}$ for all closed subclones **D** of **O**;
- **C** has automatic homeomorphicity $\Leftrightarrow$ every clone isomorphism between **C** and a closed subclone of **O** is a homeomorphism;
- **C** has automatic continuity $\Leftrightarrow$ every clone homomorphism from **C** into **O** is continuous.

Observation. (2) $\implies$ (1).

Fact. For groups (3) $\implies$ (2).

Groups: the small index property

Automorphism groups with automatic continuity:

- $(\mathbb{N}; =)$ (Dixon+Neumann+Thomas'86)
- $(\mathbb{Q}; <)$ and the atomless Boolean algebra (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)
- the random K_n-free graphs (Herwig'98)

Groups: Rubin's forall-exists interpretations

Automorphism groups with automatic homeomorphicity:

- the random graph
  $(\mathbb{Q}; <)$
  all homogeneous countable graphs
  various ω-categorical semilinear orders
  the random partial order
  the random tournament
  (Rubin '94)
- the random *k*-hypergraphs
  the Henson digraphs
  (Barbina+MacPherson '07).

Clones + monoids: Negative results

Observation

If Δ is ω-categorical,
then $\mathrm{Emb}(\Delta)$ does not have automatic continuity.

Theorem (Evans + Hewitt '90)

There exists an ω-categorical Δ such that
$\mathrm{Aut}(\Delta)$ does not have reconstruction.

Theorem (Bodirsky + Evans + Kompatscher + MP '16)

$\mathrm{Pol}(\Delta)$, $\mathrm{End}(\Delta)$, $\overline{\mathrm{Aut}(\Delta)}$ do not have reconstruction.

Method I: Automatic continuity via Birkhoff's theorem

Let **C** be a closed subclone of **O**, and $\xi\colon \mathbf{C} \to \mathbf{O}$ be a homomorphism.

Theorem (Birkhoff '35)

The algebra $(\omega; \xi[\mathbf{C}])$ is an HSP of the algebra $(\omega; \mathbf{C})$.

The only possibly discontinuous step is an infinite product.

Theorem (Bodirsky + MP + Pongrácz '13)

Any closed subclone of **O** containing ω^ω has automatic continuity and automatic homeomorphicity.

Method II: Automatic homeomorphicity via groups

Let **C** be a closed subclone of **O**
whose group $\mathbf{G}_C$ of invertibles has automatic homeomorphicity.

Show:

- the closure of $\mathbf{G}_C$ in **O** has reconstruction;
- the clone of unary functions of **C** has reconstruction;
- **C** has reconstruction.

Theorem (Bodirsky + MP + Pongrácz '13)

Let G be the random graph.
The following have automatic homeomorphicity:

- End(G);
- Pol(G);
- Various other famous clones containing Aut(G).

Method III: Rubin's interpretations

Interpret structure Δ in the algebraic structure of its clone $\mathrm{Pol}(\Delta)$.

Theorem (Maissel + Rubin '15)

Let $\mathrm{Pol}(\Delta), \mathrm{Pol}(\Delta')$ contain all transpositions on their domain ω.

Then any clone isomorphism $\mathrm{Pol}(\Delta) \rightarrow \mathrm{Pol}(\Delta')$
is induced by a permutation of ω.

Part IV

The open problem

The open problem

Let **1** be the clone containing only projections – the smallest clone.

Problem

Let Δ be ω-categorical.

- If $\mathrm{Pol}(\Delta) \to \mathbf{1}$ via a clone homomorphism, then also continuously?
- $\mathbf{1} \in \mathrm{HSP}(\mathrm{Pol}(\Delta))$ implies $\mathbf{1} \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\Delta))$?

Theorem (Barto + Kompatscher + Olšák + Van Pham + MP '17)

Let Δ be ω-categorical, with less than double exponential type growth. TFAE:

- There is no linear uniformly continuous homomorphism $\mathrm{Pol}(\Delta) \to \mathbf{1}$;
- $\mathrm{Pol}(\Delta)$ contains functions u, v (unary) and s (6-ary) such that

$$\forall x, y, z \;\; (u \circ s(x, y, x, z, y, z) = v \circ s(y, x, z, x, z, y)) \, .$$

Thank you!