Ramsey Groups

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In memory of Mati Rubin

Introduction

Some years ago, I wrote a paper, "Partitions and Permutation Groups", connecting

- permutation models of ZFA that satisfy the Boolean prime ideal theorem and
- extremely amenable topological groups.

A digression in that paper contained two questions that led to the present talk.

Definition

Let G be a group acting by permutations on a set X.

X is a *Ramsey G set* if, for every finite $A \subseteq X$ and every coloring $c : X \to 2$, there is some $g \in G$ such that c is constant on $gA = \{ga : a \in A\}$.

A Ramsey G-set is necessarily transitive and therefore isomorphic to the set G/H of cosets gH of some subgroup H of G.

Such an H is a *Ramsey subgroup* of G.

Immediate Consequences

In the definition of Ramsey G-sets, it wouldn't matter if we allowed more than 2 colors:

A G-set X is Ramsey iff, for every finite $A \subseteq X$, every finite k, and every coloring $c : X \to k$, there is some $g \in G$ such that c is constant on gA.

We can also color only a sufficiently large, finite part of X:

A G-set X is Ramsey iff, for every finite $A \subseteq X$, there is a finite $F \subseteq X$ such that, for every coloring $c : F \to 2$ there is some $g \in G$ such that $gA \subseteq F$ and c is constant on gA.

Context for the Questions

Suppose $K \leq H \leq G$ are groups and K is a Ramsey subgroup of G.

Then

- K is a Ramsey subgroup of H, and
- H is a Ramsey subgroup of G.

In other words, quotients and fibers of Ramsey G-sets are Ramsey (for the appropriate group) also.

Question 1

First question: Converse?

If K is a Ramsey subgroup of H and H a Ramsey subgroup of G, must K be a Ramsey subgroup of G?

If a G-set has a Ramsey quotient with a Ramsey fiber, must that G-set be Ramsey?

Special Case: Products

Suppose X_1 is a Ramsey G_1-set and X_2 is a Ramsey G_2-set. Then $X_1 \times X_2$ is a Ramsey $G_1 \times G_2$-set.

This is the special case of the first question with

- $G = G_1 \times G_2$,
- $H = H_1 \times G_2$, and
- $K = H_1 \times H_2$,

for some subgroups H_i of G_i, namely the stabilizers of a point in X_i.

Answer to Question 1

Dana Bartošová found a counterexample for the first question.

- G is the group of all permutations of an infinite set,
- H is the stabilizer of one point, and
- K is the pointwise stabilizer of two distinct points.

Dana also pointed out that, in view of the connection with extremely amenable groups, I should have required my groups to preserve a linear ordering.

Question 2

So I still had high hopes for my second question, because the groups there preserved a linear ordering.

Is the rational plane $\mathbb{Q}^2$ a Ramsey G-set for the group G of permutations of the form

$$(x, y) \mapsto (ax + b, \frac{y}{a} + c)$$

for $a, b, c \in \mathbb{Q}$ and $a > 0$
("special affine transformations")?

The condition $a > 0$ ensures that the group action preserves the lexicographic order on $\mathbb{Q}^2$.

Connection Between Questions

$\mathbb{Q}^2$ projects to the x-axis $\mathbb{Q}$ with the affine action

$$x \mapsto ax + b.$$

This action is Ramsey, by van der Waerden's theorem on homogeneous arithmetic progressions.

The stabilizer H of 0 in this quotient consists of the transformations where $b = 0$, i.e.,

$$(x, y) \mapsto (ax, \frac{y}{a} + c).$$

The fiber over 0 is the y-axis, on which H acts by $y \mapsto \frac{y}{a} + c$, also a Ramsey action.

So Question 2 is a special case of Question 1, with a preserved ordering.

More Reason for Hope

A minor variant of Question 2 makes the dilation factor the same in both directions, rather than reciprocals:

$$(x, y) \mapsto (ax + b, ay + c).$$

This variant has a positive answer.
It follows from the Hales-Jewett theorem by almost the same argument as van der Waerden's theorem.

Answer to Question 2

Imre Leader, Christian Reiher, and Mark Walters gave a counterexample for Question 2.

There is a 4-coloring of $\mathbb{Q}^2$ such that no set of the form

$$\{(b, c), (a + b, c), (b, \frac{1}{a} + c), (a + b, \frac{1}{a} + c)\}$$

is homogeneous. (The set consists of the four corners of a unit-area, axis-parallel, rational rectangle.)

What Can Be Salvaged?

Try to extend the proof for products to more general situations.
It works, under **strong** additional hypotheses.

Suppose that K is a Ramsey subgroup of H, that H is a Ramsey subgroup of G, and that there are subgroups $S \leq G$ and $T \leq H$ such that

- S meets every coset of every finite intersection of G-conjugates of H,
- T meets every coset of every finite intersection of H-conjugates of K,
- every element of S commutes with every element of T, and
- $S \cap H$ is a subset of every H-conjugate of K.

Then K is a Ramsey subgroup of G.

Making Assumptions Look Natural

Consider the hypothesis that S meets every coset of every finite intersection of G-conjugates of H.

Those finite intersections of conjugates are a basis for the smallest normal filter of subgroups of G that contains H.

Normal filters of subgroups are an essential ingredient in permutation models of ZFA.

They also serve as neighborhood bases at the identity when one makes G a topological group with small open subgroups.

Their cosets form a base for that topology.

So our hypothesis says that S is dense in G for the coarsest group topology making H open.

Another Special Case

Can the positive result for the $(ax + b, ay + c)$ action on $\mathbb{Q}^2$ be deduced from van der Waerden's theorem?