Automorphisms Groups and Reconstruction Problems

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Eilat, 25 April 2018

Reconstruction Theory

Reconstruction theory deals with the problem of reconstruction of model-theoretic properties of a (countable) structure M from a naturally associated algebraic or topological object $X(M)$, e.g. its automorphism group or its endomorphism monoid.

Prototipical results are:

$$X(M) \sim_1 X(M') \text{ if and only if } M \sim_2 M',$$

for given equivalence relations $\sim_1$ and $\sim_2$ of interest.

Connections with Descriptive ST and Combinatorics

This field or research was popular in the 80's/90's, and several leading model-theorists constributed to the field, e.g. Hodges, Hrushovski, Machpersohn, Rubin and Shelah.

Most recently this field was rediscovered by descriptive set-theorists (e.g. Kechris, Rosendal and Solecki) finding deep connections with finite combinatorics (based on earlier works of Hrushovski).

In this talk we will look at some recent results (both positive and negative) in the area, some applications, and suggest new possible connections with descriptive set theory.

Techniques and Degrees of Reconstruction

Two independent techniques lead the scene in this field:

(A) the (strong) small index property;

(B) Rubin's $\forall\exists$-interpretation.

Furthermore, there are two classical degrees of reconstruction:

1. reconstruction up to bi-interpretability;
2. reconstruction up to bi-definability.

These correspond to:

1. abstract group $\cong \implies$ topological group $\cong$;
2. abstract group $\cong \implies$ permutation group $\cong$.

The Strong Small Index Property

Let M be a countable structure.

- Small index property (SIP): every subgroup of $Aut(M)$ of index less than the continuum contains the pointwise stabilizer of a finite set $A \subseteq M$.
- Strong small index property (SSIP): every subgroup of $Aut(M)$ of index less than the continuum lies between the pointwise and the setwise stabilizer of a finite set $A \subseteq M$.

Bi-Definability

We consider structures in possibly different languages!

Definition

We say that two structures M and N are bi-definable if there is a bijection $f : M \to N$ such that for every $A \subseteq M^n$, A is $\emptyset$-definable in M if and only if $f(A)$ is $\emptyset$-definable in N.

Bi-Interpretability

Definition

Let M and N be models. We say that N is interpretable in M if for some $n < \omega$ there are:

(1) *a $\emptyset$-definable subset D of M^n;*

(2) *a $\emptyset$-definable equivalence relation on D;*

(3) *a bijection $\alpha : N \to D/E$ such that for every $m < \omega$ and $\emptyset$-definable subset R of N^m the subset of M^{nm} given by:*

$$\hat{R} = \{(\bar{a}_1, ..., \bar{a}_m) \in (M^n)^m : (\alpha^{-1}(\bar{a}_1/E), ..., \alpha^{-1}(\bar{a}_m/E)) \in R\}$$

is $\emptyset$-definable in M.

Definition

We say that two structures M and N are bi-interpretable if they are mutually intepretable, say via F and G, and $F \circ G$ and $G \circ F$ are $\emptyset$-definable (all this in M^{eq} and N^{eq}).

State of the Art on First Degree of Reconstruction

Matatyahu Rubin. On the Reconstruction of $\aleph_0$-Categorical Structures from their Automorphism Groups.
Proc. London Math. Soc. (3) 69 (1994), no. 2, 225-249.

Theorem (Rubin)

Let M and N be countable $\aleph_0$-categorical structures and suppose that M has a $\forall\exists$-interpretation. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-interpretable.

Theorem (Lascar (based on works of Ahlbrandt and Ziegler))

Let M and N be countable $\aleph_0$-categorical structures and suppose that M has the small index property. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-interpretable.

State of the Art on Second Degree of Reconstruction

Theorem (Rubin)

Let M and N be countable $\aleph_0$-categorical structures with no algebraicity and suppose that M has a $\forall\exists$-interpretation. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-definable.

Theorem

?

Algebraicity

Definition

Let M be a structure and $G = Aut(M)$.

(1) *We say that a is algebraic over $A \subseteq M$ in M if the orbit of a under $G_{(A)}$ is finite.*

(2) *The* algebraic closure *of $A \subseteq M$ in M, denoted as $acl_M(A)$, is the set of elements of M which are algebraic over A.*

The Missing Piece

Theorem (P. & Shelah)

Let $\mathbf{K}_*$ be the class of countable structures M satisfying:

(1) *M has the strong small index property;*

(2) *for every finite $A \subseteq M$, $acl_M(A)$ is finite;*

(3) *for every $a \in M$, $acl_M(\{a\}) = \{a\}$.*

Then for $M, N \in \mathbf{K}_*$, $Aut(M)$ and $Aut(N)$ are isomorphic as abstract groups if and only if $(Aut(M), M)$ and $(Aut(N), N)$ are isomorphic as permutation groups.

The Missing Piece (Cont.)

Corollary (P. & Shelah)

Let M and N be countable $\aleph_0$-categorical structures with the strong small index property and no algebraicity. Then $Aut(M)$ and $Aut(N)$ are isomorphic as abstract groups if and only if M and N are bi-definable. Furthermore, if $f : M \to N$ witnesses the bi-definability of M and N, then f induces the isomorphism of abstract groups $\pi_f : Aut(M) \cong Aut(N)$ given by $\alpha \mapsto f \alpha f^{-1}$.

Outer Automorphisms Groups

Given a group G we let:

$$Out(G) = Aut(G)/Inn(G).$$

We say that G is complete if it has trivial center and $Aut(G) = Inn(G)$ (which implies that $G \cong Aut(G)$).

If M satisfies the conclusion of the theorem above, then any $f \in Aut(Aut(M))$ is induced by a permutation of M. With this it is easy to see (cf. also Rubin's paper cited above):

(1) Letting R_n be the n-coloured random graph ($n \geqslant 2$) we have that $Out(Aut(R_n)) \cong Sym(n)$.

(2) Letting M_n be the K_n-free random graph ($n \geqslant 3$) we have that $Aut(M_n)$ is complete.

Outer Automorphisms Groups (Cont.)

Theorem (P. & Shelah)

Let K be a finite group. Then there exists a countable $\aleph_0$-categorical homogeneous structure M with the strong small index property and no algebraicity such that $K \cong Out(Aut(M))$.

The Crucial Point

Let M be a countable structure with SSIP.

(1) We let $\mathbf{A}(M) = \{acl_M(B) : B \subseteq_{fin} M\}$.
(2) We let $\mathbf{EA}(M) = \{(K, L) : K \in \mathbf{A}(M) \text{ and } L \leqslant Aut(K)\}$.

Let $(K, L) \in \mathbf{EA}(M)$, we define:

$$G_{(K,L)} = \{f \in Aut(M) : f \restriction K \in L\}.$$

We let:

$$\mathcal{SS}(M) = \{G_{(K,L)} : (K, L) \in \mathbf{EA}(M)\}.$$

Lemma

Let $\mathcal{G} = \{H \leqslant G : [G : H] < 2^\omega\}$. Then $\mathcal{G} = \mathcal{SS}(M)$.

The Expanded Group of Automorphisms

We define the structure $ExAut(M)$, the expanded group of automorphisms of M, as follows:

- $ExAut(M)$ is a two-sorted structure;
- the first sort has set of elements $Aut(M) = G$;
- the second sort has set of elements $\mathbf{EA}(M)$;
- we have various relations and functions relating the two sorts.

Definability of $ExAut(M)$ in $Aut(M)$

- We say that a set of subsets of a structure N is second-order definable if it is preserved by automorphisms of N.
- We say that a structure M is second-order definable in a structure N if there is a injective map $\mathbf{j}$ mapping $\emptyset$-definable subsets of M to second-order definable set of subsets of N.

Theorem

(1) *The map*

$$\mathbf{j}_M = \mathbf{j} : (f, (K, L)) \mapsto (f, G_{(K,L)})$$

witnesses second-order definability of $ExAut(M)$ in $Aut(M)$.

(2) *Every $F \in Aut(G)$ has an extension $\hat{F} \in Aut(ExAut(M))$.*

SIP, Homogeneous Structures and Finite Combinatorics

A countable structure M is said to be homogeneous if every isomorphism between finitely generated substructures of M extends to an automorphism of M.

In a famous paper from '93 Hodges, Hodkinson, Lascar and Shelah essentially reduced the problem of determination of SIP of a given countable homogeneous structure M to the following combinatorial problem (Extension Property for Partial Automorphisms) on the collection $\mathbf{K}(M) = \mathbf{K}$ of finitely generated substructures of M.

Definition (EPPA)

For every $A \in \mathbf{K}$ there exists $B \in \mathbf{K}$ such that A is a substructure of B and every partial automorphism of A extends to an automorphism of B.

Extension Property for Partial Automorphisms

In particular they showed that in order to prove that the random graph has SIP is suffices to prove the EPPA for the class of finite graphs. In a celebrated study Hushovski solved this problem, thus allowing for the conclusion that the random graph has SIP.

Since then many authors generalized Hrushovski's proof, developing a whole new branch of finite combinatorics, centred around this and related problems, e.g. Ramsey properties.

Many Structures with SSIP

Theorem (P. & Shelah)

Let M be a countable homogeneous structure with canonical amalgamation and locally finite algebraicity. If M has the small index property, then M has the strong small index property.

Corollary (P. & Shelah (based on Siniora and Solecki))

Let M be a countable free homogeneous structure in a locally finite irreflexive relational language. Then M has the strong small index property.

Corollary (P. & Shelah)

The following structures have the strong small index property:

(1) *the n-coloured random graph;*

(2) *the universal homogeneous K_n-free graph ($n \geqslant 3$);*

(3) *the random directed graph;*

(4) *the continuum many Henson digraphs.*

Many Structures with SSIP (Cont.)

We then define what we call η-hypergraphs and random ζ-free η-hypergraphs, for some $\eta \in 2^\omega$ and $\zeta \in \omega^\omega$, and show:

Theorem

Let $\mathbf{K}$ *be one of the following two classes of countable structures:*

1. *the random* η*-hypergraphs* $M(\eta)$*, for some* $\eta \in 2^\omega$*;*
2. *the random* ζ*-free* η*-hypergraphs* $M(\eta, \zeta)$*, for some* $\eta \in 2^\omega$ *and* $\zeta \in \omega^\omega$*.*

If $M = M(x)$*,* $N = N(y) \in \mathbf{K}$ *and* $x \neq y$*, then* $Aut(M) \not\cong Aut(N)$*.*

Hall's Universal Locally Finite Group

Hall's universal group is the unique countable group H such that:

- H embeds every finite group;
- any two isomorphic finite subgroups of H are conjugate in H.

Theorem (P. & Shelah)

- *$Aut(H)$ has SSIP;*
- *$Aut(H)$ is complete ($Aut(Aut(H)) = Inn(Aut(H))$).*

Hall's Universal Locally Finite Group (Cont.)

Theorem

Inn(H) is the locally finite radical of Aut(H) (i.e. it is the largest locally finite normal subgroup of Aut(H)).

Theorem

For every countable locally finite G there exists $G \cong G' \leqslant H$ such that every $f \in Aut(G')$ extends to an $\hat{f} \in Aut(H)$ in such a way that $f \mapsto \hat{f}$ embeds $Aut(G')$ into $Aut(H)$.

In particular, we solved three open questions of Hickin on $Aut(H)$, and gave a partial answer to Question VI.5 of Kegel and Wehrfritz from their book on locally finite groups.

Automorphism Groups and Model-Theoretic Stability

One of the main tools in modern model theory are certain dividing lines which classify theories in function of how much combinatorial information can be coded in the class of its models.

This area of research was invented by Shelah and has had huge number of applications in the most disparate fields of mathematics.

The most well-known dividing lines are: $\aleph_0$-stability, superstability, and stability. Each of these notions can be defined in terms of bounds on the number of types over sets of parameters for models of a given theory, but there are many equivalent definitions.

On a Problem of Rosendal

In a recent work Rosendal isolates a property of topological groups which he calls locally boundedness and proves that if M is the countable, saturated model of an $\aleph_0$-stable theory then $Aut(M)$ is locally bounded.

Rosendal asks if the property of locally boundedness is satisfied by the group of automorphisms of any countable model of an $\aleph_0$-stable theory. This was settled in the negative by Zielinski.

An Impossibility Result

Theorem (P. & Shelah)

For every countable structure M there exists an $\aleph_0$-stable (with NDOP and NOTOP) countable structure N such that $Aut(M)$ and $Aut(N)$ are topologically isomorphic with respect to the naturally associated Polish group topologies.

This shows that it is impossible to detect any form of stability of a countable structure M from the topological properties of the Polish group $Aut(M)$.

New Directions

We conclude the talk proposing some new directions relating reconstruction theory with invariant descriptive set theory.

Specifically, we propose a descriptive set theoretic analysis of the following model-theoretic equivalence relations:

- $M \sim_1 M'$ if and only if M and M' are bi-definable;
- $M \sim_2 M'$ if and only if M and M' are mutually interpretable;
- $M \sim_3 M'$ if and only if M and M' are bi-interpretable;
- $M \sim_4 M'$ if and only if $Aut(M) \cong Aut(M')$.

It is well know that the isomorphism relation on Borel classes of countable structures is never complete analytic. Are any of the relations $\sim_i$ above complete analytic? What is their relation in terms of Borel complexity of equivalence relations?

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