On isomorphism of κ-dense sets of reals and related problems

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23 April 2018, Eilat

Set Theory, Model Theory and Applications
in memoriam Mati Rubin

Theorem (Cantor, 1895)**.**
1. All countable linearly ordered dense sets with no end-points are isomorphic.
2. Any linearly ordered dense set is $\aleph_0$-universal.

Here: a linearly ordered set X is κ-*universal* iff any linearly ordered set of size $\leq \kappa$ can be embedded into X.

Thus $\mathbb{Q}$, the set of rationals with their usual order, is $\aleph_0$-universal, and is a unique (up to isomorphism) countable dense set with no end-ponts.

Hausdorff generalized these results from $\aleph_0$ to all cardinals $\aleph_\alpha$.

A linearly ordered set X is called an η_α-*set* iff for every $A, B \in \mathcal{P}_{\aleph_\alpha}(X)$ with $A < B$ (for all elements) there exists $x \in X$ with $A < x < B$.

Facts. Let X be a linearly ordered set.
1. X is an η_0-set iff X is dense with no end-points.
2. If X is an η_α-set, then $|X| \geq \aleph_\alpha$ and X has no end-points.

Theorem (Hausdorff, 1907)**.**
1. Any η_α*-set is* $\aleph_\alpha$*-universal.*
2. If $2^{<\aleph_\alpha} = \aleph_\alpha = \mathrm{cf}\,\aleph_\alpha$*, then the set* $(2^{<\omega_\alpha}, <_{\mathrm{lex}})$ *is an* η_α*-set of size* $\aleph_\alpha$*.*
3. All η_α*-sets of size* $\aleph_\alpha$ *are isomorphic.*

Here:
$X^{<\alpha}$ denotes the set $\bigcup_{\beta<\alpha} X^\beta$,
$\kappa^{<\lambda}$ denotes the cardinal $\sum_{\mu<\lambda} \kappa^\mu$.

Remark (Mendelson, 1958)**.** The set $(2^{<\kappa}, <_{\mathrm{lex}})$ is κ-universal.

The η_α-sets are prototypical instances of saturated and universal models; and saturated models are universal.

X is κ-*saturated* iff for all $A \in \mathcal{P}_\kappa(X)$, X realizes all complete types over A consistent with the theory of X with constants in A.

Theorem. *Let X be a linearly ordered dense set without end-points. Then X is an η_α-set iff X is $\aleph_\alpha$-saturated.*

Cantor's theorem can be generalized in another direction:

Theorem (Baumgartner, 1971). *It is consistent w.r.t.* ZFC *that all* $\aleph_1$*-dense sets of reals are isomorphic.*

Here: a linearly ordered set X is κ-*dense* iff it has no end-points and any its non-trivial interval has size κ. In particular, $|X| = \kappa$.

Let BA denote the conclusion of Baumgartner's theorem.

Idea of proof: Under CH, for $\aleph_1$-dense $A, B \subseteq \mathbb{R}$ there is a c.c.c. forcing making A and B isomorphic. Then iterate.

Starting from $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, it goes to $\mathrm{BA} + (2^{\aleph_0} = \aleph_2)$.

Facts. *Under* BA, *we have:*
1. $2^{\aleph_0} = 2^{\aleph_1}$.
2. All sets of reals of size $< \aleph_2$ *are universally measure zero sets and universally meager.*

Here: X is a *universally measure zero* (*universally meager*) set iff any Borel isomorphic image of X has Lebesgue measure zero (is meager).

Proof:
1. There exist: $2^{\aleph_1}$ sets of reals that are $\aleph_1$-dense and $2^{\aleph_0}$ order isomorphisms of the real line.
2. There is such a set of real of size $\aleph_1$ (Sierpiński, 1934).

Baumgartner posed two questions. One of them: *Does* MA *imply* BA*?*

Avraham, Shelah, 1981: No. Some new methods and other results, e.g., consistency of "Any function of $\mathbb{R}$ into itself is monotone on an uncountable set".

Theorem (Abraham, Rubin, Shelah, 1985)**.** *For any finite poset L, TFAE:*
(i) MA+ *"the set of order types of $\aleph_1$-dense homogeneous sets of reals with the relation of embeddability is isomorphic to L",*
(ii) L is a distributive lattice.

Also: $\text{BA} + 2^{\aleph_0} > \aleph_2$ is consistent (w.r.t. ZFC).

Let BA_κ denote the assertion "All κ-dense sets of reals are isomorphic".

Fact. BA_κ *implies* $2^\kappa = 2^{\aleph_0}$.

Baumgartner's 2nd question: *Is* $\mathrm{BA}_{\aleph_2}$ *consistent?*

To the best of my knowledge, answered recently:

Theorem (Neeman, Moore–Todorcevic, 2017)**.** *It is possible to have* $\mathrm{BA}_{\aleph_1} + \mathrm{BA}_{\aleph_2}$.

Theorem (D.S., 2001). *The set* $\{\kappa : \mathsf{BA}_\kappa\}$ *has the suprema of all its bounded countable subsets. Moreover, if* BA_{κ_n} *for all* $n < \omega$, *and* $\kappa < \mathfrak{q}_0$ *where* $\kappa = \sup_{n<\omega} \kappa_n$, *then* BA_κ.

Here: $\mathfrak{q}_0 = \min\{|X| : X \subseteq \mathbb{R}$ is not a Q-set$\}$,
X is a Q-*set* iff all its subsets are F_σ (and G_δ).

Fact. All Q-sets are meager and null. Moreover, they are in $\mathcal{E}$, the σ-ideal generated by closed null sets of reals, i.e. $\mathfrak{q}_0 \leq \mathrm{non}\,(\mathcal{E})$.

Idea of proof:

Lemma. Under BA_κ, all $X \in \mathcal{P}_{\kappa^+}(\mathbb{R})$ are Q-sets, i.e. $\min\{\kappa : \neg\,\mathsf{BA}_\kappa\} \le \mathfrak{q}_0$.

Suppose $X, Y \in [\mathbb{R}]^\kappa$ and $\kappa = \lim_{n\to\omega} \kappa_n < \mathfrak{q}_0$. Decompose X and Y, in a certain regular way, into two increasing sequences of subsets, X_n and Y_n, such that:

(i) $|X| = |Y| = \kappa_n$,
(ii) X_n and Y_n are closed nowhere dense subsets of X_{n+1} and Y_{n+1}, resp.,

and construct isomorphisms $f_n : X_n \to Y_n$ such that $f_n \subseteq f_{n+1}$. Then $\bigcup_n f_n$ is an isomorphism of X and Y.

(The fact that X and Y are Q-sets is used in (ii). Also choosing of partial isomorphisms should be controlled.)

Define
$\mathfrak{j}_0 = \min\{\kappa : \neg\,\mathsf{BA}_\kappa\}$,
$\mathfrak{j} = \min\{\kappa : \neg\,\mathsf{BA}_\lambda \text{ for all } \lambda \geq \kappa\}$.

Corollary.
1. $\mathfrak{j}_0 \leq \mathfrak{j} \leq \mathfrak{q}_0 \leq \mathrm{non}\,(\mathcal{E})$.
2. If $\mathrm{cf}\,\mathfrak{q}_0 > \omega$, *then* $\mathrm{cf}\,\mathfrak{j}_0 > \omega$ *and* $\mathrm{cf}\,\mathfrak{j} > \omega$.

Questions.

1. Can we have $\mathfrak{j}_0 < \mathfrak{j}$? i.e. $\neg\,\mathsf{BA}_\kappa + \mathsf{BA}_\lambda$ for some $\kappa < \lambda$. (Conjecture: Yes. E.g. $\neg\,\mathsf{BA}_{\omega_1} + \mathsf{BA}_{\omega_2}$.)

2. Can we have $\mathrm{cf}\,\mathfrak{j}_0 = \omega$? $\mathrm{cf}\,\mathfrak{j} = \omega$? $\mathrm{cf}\,\mathfrak{q}_0 = \omega$? (Conjecture: $\mathrm{cf}\,\mathfrak{q}_0 = \omega$ is consistent. Cf. Fleissner–Miller, 1980: there can be $A, B \subseteq \mathbb{R}$ such that A is a Q-set, $|B| = \aleph_0$, and $A \cup B$ is not a Q-set.)

The same idea of proof can be used to get

Theorem (D.S., 2001). *All* $2^{\aleph_0}$*-dense meager* F_σ*-sets of reals are isomorphic.*

(This also can be considered as a counterpart of Cantor's theorem.)

Actually, both theorems follow from an abstract theorem. Let us call a linearly ordered set without end-points *self-similar* iff it is isomorphic to each its non-trivial interval.

Theorem. *Let linearly ordered sets* X *and* Y *be such that:*

(i) X *is decomposed into an increasing sequence of dense subsets* X_n*,* $n < \omega$*, which are self-similar, meager and relatively* F_σ*,*
(ii) Y *is decomposed into an increasing sequence of subsets* Y_n*,* $n < \omega$*, such that* X_n *and* Y_n *are isomorphic.*

Then X *and* Y *are isomorphic and self-similar.*

Corollary. *Any* $2^{\aleph_0}$*-dense meager* F_σ*-set of reals is self-similar and isomorphic to its reverse.*

Corollary. *There are exactly two order-types of* G_δ*-sets of reals that are dense and nowhere open:*

(i) the order-type of irrationals, and
(ii) the order-type of the complement of any $2^{\aleph_0}$*-dense meager* F_σ*-set of reals.*

Cf.: there is exactly one homeomorphism type of such sets (from Alexandroff–Hausdorff's theorem). This fact shows that the order classification is finer than the topological one.

Let $\mathsf{BA}_\kappa^{\mathsf{hom}}$ denote the assertion "All κ-dense sets of reals are homeomorphic". Clearly, BA_κ implies $\mathsf{BA}_\kappa^{\mathsf{hom}}$.

Questions.

1. Can we have $\mathsf{BA}_\kappa^{\mathsf{hom}} + \neg\,\mathsf{BA}_\kappa$?

2. Is the set of Borel order-types quasi-well-founded by embedding? (Cf. Laver's result on scattered order types.)

3. Classify order-types of homogeneous (or self-similar) sets in $\Sigma^0_\alpha \setminus \Pi^0_\alpha$.

Likewise for other definable sets of reals.

What is a model-theoretic side of BA, its variants and generalizations?

Let $\mathsf{BA}_\lambda(X,<)$ denote the assertion "All λ-dense subsets of $(X,<)$ are isomorphic".

Questions.

1. Let $2^{<\kappa} = \kappa = \mathrm{cf}\,\kappa$ (for simplicity). Then we have $\mathsf{BA}_\kappa(\kappa^\kappa, <_{\mathsf{lex}})$ by Hausdorff's generalization of Cantor's theorem. Can we have $\mathsf{BA}_\lambda(\kappa^\kappa, <_{\mathsf{lex}})$ for $\lambda > \kappa$? (Conjecture: Yes.)

2. Can we have all $\aleph_1$-dense subalgebras of the Cohen algebra (= the Dedekind–MacNeill completion of the atomless countable Boolean algebra) isomorphic? (Use Baumgartner's argument.)

3. Versions of categoricity and saturatedness in a class of models.