On Rubin's 'Theories of linear order'

Predrag Tanović

Mathematical Institute SANU and
Faculty of Mathematics, University of Belgrade, Serbia

Set Theory, Model Theory and Applications
(in memory of Mati Rubin)
Eilat, April 2018.

Rubin's 1973 Master's Thesis

Theories of linear order. Israel Journal of Mathematics, March 1974, Volume 17, Issue 4, pp 392–443

- Made a deep analysis of $L_{\omega,\omega}$-theories of coloured orders (linear orders with unary predicates);
- Understood the impact of naming a single parameter on 1-types: extensions of $p \in S_1(T)$ in $S_1(a)$;
- $(S_1(T))^n$ vs $S_n(T)$ (T is binary, so $n = 2$ suffices).

Consequences of his analysis

T - complete, countable theory of coloured orders.

1. The number of countable models:
   – either finite or $2^{\aleph_0}$;
   – either 1 or $2^{\aleph_0}$ if L is finite.
2. If $|S_1(T)| \leqslant \aleph_0$, then $|S_n(T)| \leqslant \aleph_0$ for all n;
3. If $CB(S_1(T)) < \nu$, then $CB(S_2(T)) < \nu^4 \cdot 4 + 20$;
4. If $S_1(T)$ is finite, then T is finitely axiomatizable.

Questions:

1. What are the key model-theoretical properties of coloured orders (in Rubin's work)?

2. Is there a 'geometric' description of definable sets?

3. Can we count countable models (Vaught's conjecture) in some wider, syntax-free context for T?

From now on work in a saturated linearly ordered L-structure $(\mathbb{U}, <, \ldots)$ (any L); T its complete theory.

The key property of coloured orders...

... was extracted from Rubin's work by Bruno Poizat in his first book...

- A sequence of formulas $\phi_1(x), \dots, \phi_n(x)$ is realized between a and b if there are elements $a < c_1 < \dots < c_n < b$ with each c_i satisfying $\phi_i(x)$.

Rubin's binarity

(RB) Two increasing n-tuples $a_1 < \dots < a_n$ and $b_1 < \dots < b_n$ of elements of $\mathbb{U}$ have the same type if and only if they satisfy the following conditions:

- $\mathrm{tp}(a_i) = \mathrm{tp}(b_i)$ for every $i \leqslant n$;
- For every $i \leqslant n$ the same finite sequences of formulas are realized between a_i and a_{i+1} as between b_i and b_{i+1}.

- Poizat's Theorem 12.32, which he calls Rubin's Theorem, is a restatement of Rubin's Lemma 3.5 and states that coloured orders satisfy (RB).

Rubin's Theorem (both directions)

Condition (RB) characterizes coloured orders up to definitional equivalence (among saturated, linearly ordered structures).

(RB) is not geometric, so ...

A geometric description

Theorem(Pierre Simon, 2009)

A coloured order expanded by naming all definable unary sets and monotone relations, eliminates quantifiers.

- $R \subseteq A \times B$ is *monotone* if $x \leqslant_A x' \, R y' \leqslant_B y$ implies $x \, R \, y$. (Or: fibers $(R(A, b) \mid b \in B)$ are initial parts and $\subseteq$-increase.)

Corollary

Coloured orders are *weakly quasi o-minimal* structures: every $L(A)$-definable set of singletons is a Boolean combination of unary L-definable sets and $L(A)$-definable convex sets.

Simon uses Rubin's Lemma 7.9:

Linear binarity

(LB) For all increasing sequences $a_1 < a_2 < \ldots < a_n$:

$$\cup_{1\leqslant i<n} \mathrm{tp}_{x_i,x_{i+1}}(a_i, a_{i+1}) \vdash \mathrm{tp}_{x_1,\ldots,x_n}(a_1, \ldots, a_n).$$

Equivalents of (LB):

(1) For all $a \in \mathbb{U}$ and $f \in Aut(\mathbb{U})$ fixing a, the mapping:

$$g(x) = f(x) \text{ for } x \leqslant a \quad \text{and} \quad g(x) = x \text{ for } a < x,$$

is an automorphism of $\mathbb{U}$.

(2) $\bar{b} < a < \bar{c}$ implies $\mathrm{tp}(\bar{b}/a) \cup \mathrm{tp}(\bar{c}/a) \vdash \mathrm{tp}(\bar{b}\bar{c}/a)$

Strong linear binarity

(SLB) For every convex set $C \subset \mathbb{U}$ and $f \in Aut(\mathbb{U})$ fixing C setwise, the mapping defined by:

$$g(x) = f(x) \text{ for } x \in C \text{ and } g(x) = x \text{ for } x \notin C,$$

is an automorphism of $\mathbb{U}$.

Linear finiteness

(LF) For every convex set $C \subset \mathbb{U}$ and formula $\phi(\bar{x};\bar{y})$, there are only finitely many ϕ-types with parameters from C that are realized in $\mathbb{U} \setminus C$.

- (LF) equivalent:

$$\bar{b} < a < \bar{c} \quad \text{implies} \quad \mathrm{stp}(\bar{b}/a) \cup \mathrm{stp}(\bar{c}/a) \vdash \mathrm{tp}(\bar{b}\bar{c}/a).$$

Proposition

(RB) $\Rightarrow$ (SLB) $\Rightarrow$ (LB) $\Rightarrow$(LF).

- All implications are irreversible.

Example

$T_n = Th(\mathbb{Q}, <, E_n)$ where E_n is an equivalence relation with n dense classes.

1. T_2 satisfies (LB) $+ \neg$(SLB)
2. $T_3, T_4, \ldots$ satisfy (LF) $+ \neg$(LB)
3. One can add structure to $\mathrm{acl}^{eq}(0)$ in T_4, preserve (LF) and destroy binarity.

Coloured orders with convex equivalences

- A *ccel-order* is a coloured order expanded by convex equivalence relations.
- For each definable, convex equivalence E and an integer N we fix formulas:
  1. $S_E^N(x,y)$ describing for $N>0$ ($N<0$) that y is in the N-th consecutive E-class succeeding (preceeding) $[x]_E$; $S_E^0(x,y)$ is $E(x,y)$;
  2. $x \leqslant S_E^N(y) := \exists z\, S_E^N(y,z) \wedge \exists z\,(S_E^N(y,z) \Rightarrow x \leqslant z)$;
  3. $x < S_E^N(y)$, $S_E^N(y) < x$ and $S_E^N(y) \leqslant x$ are defined similarly.

Joint work with Slavko Moconja and Dejan Ilić.

Theorem

(SLB) describes ccel-orders up to definitional equivalence.

Theorem

(SLB) every formula is equivalent to a Boolean combination of

$$\phi(x),\ \ x \leqslant S_E^n(y),\ \ x < S_E^n(y),\ \ S_E^n(y) < x,\ \ S_E^n(y) \leqslant x;$$

(all definable convex E named, all $n \in \mathbb{Z}$).

Conclusion:

- Linear orders $\approx$ coloured orders $\approx$ ccel-orders ;
- These are the 'simplest' classes of ordered structures.

For (LF) theories we have:

- Precise description of convex sets: A-definable convex set is defined by conjunction of some $\phi(x)$ and at most two formulas
$$x < S_E^m(a) \quad \text{(or similar)}$$
for $a \in A$, $m \in \mathbb{Z}$ and definable, convex equivalences E.

- Not-so-precise description of parametrically definable sets: Every definable with parameters subset of $\mathbb{U}$ is a Boolean combination of unary L-definable sets and classes of almost convex equivalences.

Vaught's Conjecture

Natural context to generalize Rubin's result (and Laura Mayer's 1988 proof in the o-minimal case):

T-binary, weakly quasi o-minimal.

- (Mayer) o-minimal theory has $3^n \cdot 6^m$ or $2^{\aleph_0}$ countable models.
- (Alibek, Baizhanov) There is a binary, weakly o-minimal theory with $\aleph_0$ countable models.

(joint with Slavko Moconja) Stationary ordered types and the number of countable models. arXiv:1804.07231

Theorem

Vaught's conjecture is true for binary, stationary ordered theories.

- $p \in S_1(T)$ is *stationary ordered* if there is a definable linear order such that for all parametrically definable D: one of D and D^c is right-eventual on $p(\mathbb{U})$ and one of them is left-eventual.
- T is *stationary ordered* if every $p \in S_1(T)$ is so.

Stationary ordered theory admits 0-definable linear ordering.

Few technicalities from the proof

The ideas of our proof essentially differ from Rubin's proof of VC.

(Richard Rast recently gave a new proof based on Rubin's ideas arXiv:1504.03037)

1. Forking ($\not\perp$) is an equivalence relation;
2. When restricted to the locus of $p \in S_1(T)$ in a model $M \models T$, $\not\perp$ is a convex equivalence. The order-type of classes is $Inv_p(M)$, the p-invariant of M ($\sim$dim$_p$(M) in stable theories);
3. Invariants behave very well under $\not\perp^f$ (of convex types): equal or reverse.
4. Under $\not\perp^w + \perp^f$ they have isomorphic or anti-isomorphic Dedekind completions; they are 'shuffled'.

T–countable, binary, stationary ordered theory

(a) $I(\aleph_0, T) = 2^{\aleph_0}$ provided that at least one of the following conditions holds:
   (1) T is not small;
   (2) there is a non-convex type $p \in S_1(T)$;
   (3) there is a non-simple type $p \in S_1(T)$;
   (4) there are infinitely many $\not\perp^w$-classes of non-isolated types in $S_1(T)$;
   (5) there is a non-isolated forking extension of some $p \in S_1(T)$ over an 1-element domain.

(b) $I(\aleph_0, T) = \aleph_0$ iff none of the above holds and there are infinitely many $\not\perp^f$-classes of non-isolated types in $S_1(T)$;

(c) $I(\aleph_0, T) < \aleph_0$ iff none of the above holds and there are finitely many $\not\perp^f$-classes of non-isolated types in $S_1(T)$.

THANK YOU !