# Guessing models and the approachability ideal

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Set theory, model theory, and applications

**in memory of Mati Rubin**

Eilat, April 25 2018

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# Outline

1. Introduction
2. Ingredients
3. Main forcing

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# Background and history

This is joint work with my PhD student **R. Mohammadpour**.

IN 2011 C. Weiß introduced some combinatorial principles that capture the essence of large cardinals, but can hold at small cardinals.
Parameters: a regular uncountable $\kappa$ and $\lambda \geq \kappa$.
Examples : $\mathrm{TP}(\kappa, \lambda)$, $\mathrm{ITP}(\kappa, \lambda)$, $\mathrm{ISP}(\kappa, \lambda)$.
We write $P(\kappa)$ if $P(\kappa, \lambda)$ holds, for all $\lambda \geq \kappa$.

Can be thought of as two cardinal versions of the tree property. Using them, we can reformulate some classical results from the 1970s.

**Theorem**

*Let $\kappa$ be an inaccessible cardinal.*

1. [Jech] *$\kappa$ is strongly compact iff $\mathrm{TP}(\kappa)$ holds.*
2. [Magidor] *$\kappa$ is supercompact iff $\mathrm{ITP}(\kappa)$ holds.*

However, $\mathrm{TP}(\kappa)$, $\mathrm{ITP}(\kappa)$, $\mathrm{ISP}(\kappa)$ can also hold at $\omega_2$, $\omega_3$, etc.

The most striking application of these principles is the following **quasi consistency** result.

**Theorem (Viale, Weiß (2011))**

*Any standard forcing construction of a model of the Proper Forcing Axiom (PFA) requires at least a strongly compact cardinal.*

One important concept that emerged is that of a **guessing model**.

# Guessing models

The following notion was introduced by J. Hamkins.

**Definition (Approximation property)**

*Let $\kappa$ be a cardinal. Suppose $M \subseteq N$ are models of a fragment of ZFC. We say that $(M, N)$ satisfies the $\kappa$-**approximation property** if, for all $\alpha \in M$ and $X \subseteq \alpha$ with $X \in N$, if $X \cap Z \in M$, for all $Z \in M$ with $|Z|^M < \kappa$, then $X \in M$.*

This is a reformulation of the original definition.

**Definition**

*Let $\kappa$ be a cardinal and $M$ a model of a fragment of ZFC. We say that $M$ is a $\kappa$-**guessing model** if $M \cap \kappa^+ \in \kappa^+$ and $(\bar{M}, V)$ has the $\kappa$-approximation property.*

Here $\bar{M}$ denotes the Mostowski collapse of $M$.

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# Guessing models

**Definition** ($\mathrm{ISP}(\kappa^+)$)

*Let $\kappa$ be a regular cardinal. $\mathrm{ISP}(\kappa^+)$ asserts that for all $\theta$ big enough the set of $\kappa$-guessing elementary submodels of $H_\theta$ is stationary in $\mathcal{P}_{\kappa^+}(H_\theta)$.*

We may require that the guessing models have some additional properties, such as **internally unbounded**, **internally stationary**, **internally club**, etc. There are also a number of weaker versions where we only guess certain special sets or functions. These variations were systematically studied by Viale and later by Cox, Kruger, Trang, and others.

**Exercise**

Show that $\mathrm{ISP}(\kappa^+)$ implies the tree property at $\kappa^+$.

# Approachability ideal

**Definition**

*Let $\lambda$ be a regular cardinal and $\bar{a} = (a_\xi : \xi < \lambda)$ a sequence of bounded subsets of $\lambda$. We let $B(\bar{a})$ denote the set of all $\delta < \lambda$ such that there is a cofinal $c \subseteq \delta$ such that:*

1. $\mathrm{otp}(c) < \delta$*, in particular $\delta$ is singular,*
2. *for all $\gamma < \delta$, there is $\eta < \delta$ such that $c \cap \gamma = a_\eta$.*

**Definition**

*Suppose $\lambda$ is regular. $I[\lambda]$ is the ideal generated by the sets $B(\bar{a})$, for sequences $\bar{a}$ as above, and the non stationary ideal $\mathrm{NS}_\lambda$.*

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# Approachability ideal

This ideal was defined by Shelah in the late 1970s. $I[\lambda]$ and its variations have been extensively studied over the past 40 years.

For regular $\kappa < \lambda$ we let $S^{\kappa}_{\lambda} = \{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$.

**Theorem (Shelah)**

*Suppose $\lambda$ is a regular cardinal.*

1. *Then $S^{<\lambda}_{\lambda^+} \in I[\lambda^+]$.*
2. *Suppose $\kappa$ is regular and $\kappa^+ < \lambda$. Then there is a stationary subset of $S^{\kappa}_{\lambda}$ which belongs to $I[\lambda]$.*

# Approachability ideal

**Question (Shelah)**

Is it consistent to have a regular cardinal $\kappa$ such that $I[\kappa^+] \restriction S^{\kappa}_{\kappa^+}$ is the nonstationary ideal on $S^{\kappa}_{\kappa^+}$?

**Theorem (Mitchell)**

*Suppose $\kappa$ is $\kappa^+$-Mahlo. Then there is a generic extension in which $\kappa = \omega_2$ and $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ is the non stationary ideal on $S^{\omega_1}_{\omega_2}$.*

**Remark**

In Mitchell's model $\omega_3 \in I[\omega_3]$. It is not known if one can have Mitchell's result for two consecutive cardinals, say $\omega_2$ and $\omega_3$.

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# Approachability ideal

What is the connection between $\mathrm{ISP}(\kappa^+)$ and Mitchell's result?

**Fact**

*Suppose* $\bar{a} = (a_\xi : \xi < \omega_2)$ *is a sequence of bounded subsets of* $\omega_2$. *Let* $M \prec H_\theta$ *be an* $\omega_1$*-guessing model of size* $\omega_1$ *such that* $\bar{a} \in M$. *Then* $M \cap \omega_2 \notin B(\bar{a})$.

Therefore, $\mathrm{ISP}(\omega_2)$ implies that $S^{\omega_1}_{\omega_2} \notin I[\omega_2]$. However, it does not imply Mitchell's result.

**Definition** ($\mathrm{FS}(\kappa^+)$)

*Suppose* $\kappa$ *is regular.* $\mathrm{FS}(\kappa^+)$ *asserts that for all* $X \in H_{\kappa^+}$ *there is a collection* $\mathcal{G}$ *of* $\kappa$*-guessing models containing* $X$ *such that* $\{M \cap \kappa^+ : M \in \mathcal{G}\}$ *is* $\kappa$*-closed and unbounded in* $\kappa^+$.

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# Approachability ideal

**Fact**

$\mathrm{FS}(\kappa^+)$ *implies that* $I[\kappa^+] \restriction S^{\kappa}_{\kappa^+}$ *is the non stationary ideal on* $S^{\kappa}_{\kappa^+}$.

**Theorem**

*Suppose* $\kappa < \lambda$ *are supercompact cardinals. Then there is a generic extension in which* $\mathrm{ISP}(\omega_2)$, $\mathrm{ISP}(\omega_3)$ *and* $\mathrm{FS}(\omega_2)$ *hold simultaneously.*

In this model we have $2^{\omega} = 2^{\omega_1} = 2^{\omega_2} = \omega_3$, the strong tree property holds for $\omega_2$ and $\omega_3$. We can get a version of this model at higher cardinals, but it is open whether one can have $\mathrm{FS}(\omega_2)$ and $\mathrm{FS}(\omega_3)$ at the same time.

# Origin of this work

Inspired by Mitchell's breakthrough result, Neeman introduced a method for iterating proper forcing using finite chaines of elementary submodels as side conditions. This gives a new proof of the consistency of PFA, but also a possibility of getting higher cardinal versions of forcing axioms. Need to use two types of models as side conditions: countable and size $\aleph_1$ internally club (I.C.) models.

I extended this method to semiproper forcing. This lead to the notion of **virtual models**. In this work we use two types of virtual models as side conditions, but instead of I.C. models of size $\aleph_1$ we use models with much stronger closure properties that we call **Magidor models**.

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# Strong properness

**Definition (Mitchell)**

*Suppose $\mathbb{P}$ is a forcing notion and $A$ is a set. We say that $p \in \mathbb{P}$ is $(A, \mathbb{P})$-**strongly generic** if for all $q \leq p$ there is $q \restriction A \in A \cap \mathbb{P}$ such that any $r \leq q \restriction A$ with $r \in A$ is compatible with $q$.*

**Definition (Strong properness)**

*Let $\mathbb{P}$ be a forcing notion and $\mathcal{S}$ a collection of sets. We say that $\mathbb{P}$ is $\mathcal{S}$-**strongly proper** if, for all $A \in \mathcal{S}$ and $p \in A \cap \mathbb{P}$, there is $q \leq p$ that is $(A, \mathbb{P})$-strongly generic.*

**Proposition (Key fact)**

*Let $\kappa$ be regular. Suppose $\mathbb{P}$ is $\mathcal{S}$-strongly proper, for some stationary subset $\mathcal{S}$ of $\mathcal{P}_\kappa(\mathbb{P})$. If $G$ is $V$-generic over $\mathbb{P}$, then $(V, V[G])$ has the $\kappa$-approximation property.*

# Supercompact cardinals

**Theorem (Magidor)**

*The following are equivalent for a regular cardinal $\kappa$.*

1. *$\kappa$ is supercompact.*
2. *for every $\gamma > \kappa$ and $x \in V_\gamma$ there are $\bar{\kappa} < \bar{\gamma} < \kappa$ and elementary embedding $j : V_{\bar{\gamma}} \to V_\gamma$ with critical point $\bar{\kappa}$ such that $j(\bar{\kappa}) = \kappa$ and $x \in j[V_{\bar{\gamma}}]$.*

Throughout, we fix a supercompact cardinal $\kappa$.

**Definition (Magidor models)**

*We say that $M$ is a **Magidor model** if, letting $\overline{M}$ be its transitive collapse, and $\pi$ the collapsing map, we have $\overline{M} = V_{\bar{\gamma}}$, for some $\bar{\gamma} < \kappa$ with $\mathrm{cof}(\bar{\gamma}) \geq \pi(\kappa)$, and $V_{\pi(\kappa)} \subseteq M$.*

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# Virtual models

**Definition**

*We say that $A$ is* ***admissible*** *if it is transitive, $\kappa \in A$ and $(A, \in) \models \mathrm{ZFC}$.*

If $\alpha \in \mathrm{ORD}^A$ let $A_\alpha = A \cap V_\alpha$. Let

$$E_A = \{\alpha \in A : \kappa \in A_\alpha \prec A\}.$$

Note that $E_A$ is a closed, possibly empty subset of $\mathrm{ORD}^A$.
For $\alpha \in A$ let $\mathrm{next}_A(\alpha) = \min(E_A \setminus (\alpha + 1))$.

# Virtual models

**Definition**

*Suppose $A$ is admissible, $M \prec A$. and $X \subseteq A$. Let*

$$\mathrm{Hull}(M, X) = \{f(\bar{x}) : f \in M, \bar{x} \in X^{<\omega}, \bar{x} \in dom(f)\}.$$

Note that $\mathrm{Hull}(M, X)$ is the Skolem hull of $M$ and $X \cap A_\delta$, where $\delta = \sup(M \cap \mathrm{ORD})$. The point is that we don't need to know $A$ to compute $\mathrm{Hull}(M, X)$.

Now, fix an inaccessible $\lambda > \kappa$, and write $E$ for $E_{V_\lambda}$, $\mathrm{next}(\alpha)$ for $\mathrm{next}_{V_\lambda}(\alpha)$, etc.

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# Virtual models

**Definition**

*Suppose $\alpha \in E$. Let $\mathscr{A}_\alpha$ denote the set of all admissible $A$ such that $V_\alpha \prec A$ and $|A| = |V_\alpha|$.*

Note that $\mathscr{A}_\alpha$ is uniformly definable in $V_\lambda$ with parameters $\kappa$ and $\alpha$.

**Definition**

- *Let $\mathcal{C}_\alpha$ consist of all countable models $M$ such that* $\mathrm{Hull}(M, V_\alpha) \in \mathscr{A}_\alpha$.
- *Let $\mathcal{U}_\alpha$ consist of all Magidor models $N$ such that* $\mathrm{Hull}(N, V_\alpha) \in \mathscr{A}_\alpha$.

We call the models in $\mathcal{V}_\alpha = \mathcal{C}_\alpha \cup \mathcal{U}_\alpha$ the **$\alpha$-models**. We let $\eta(M)$ denote the unique $\alpha$ such that $M \in \mathcal{V}_\alpha$.

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We define what it means for two virtual models to be $\alpha$-**isomorphic**.

**Definition**

*$M \simeq_\alpha N$ if there is an isomorphism $\sigma : \mathrm{Hull}(M, V_\alpha) \to \mathrm{Hull}(N, V_\alpha)$ such that $\sigma[M] = N$.*

Clearly, $\simeq_\alpha$ is an equivalence relation. Notice that if $\alpha < \beta$ and $M \simeq_\beta N$ then $M \simeq_\alpha N$. Note that each $\simeq_\alpha$-equivalence class has a canonical representative.

**Definition**

*Suppose $\alpha, \beta \in E$ with $\alpha < \beta$, and $M$ is a $\beta$-model. Let $\overline{\mathrm{Hull}(M, V_\alpha)}$ be the Mostowski collapse of $\mathrm{Hull}(M, V_\alpha)$ and let $\pi$ be the collapsing map. Let $M \upharpoonright \alpha = \pi[M]$.*

We also need a version of the $\in$-relation at all $\alpha \in E$.

**Definition**

*Suppose $M, N$ are virtual models and $\alpha \in E$. Let $M \in_\alpha N$ if there is $M' \in N$ such that $M' \simeq_\alpha M$.*

**Remark**

If $M \subseteq V_\alpha$ and $M \in_\alpha N$ then $M \in N$. In general we could have $M \in_\alpha N$ even if the rank of $M$ is higher than the rank of $N$.

**Fact**

*Suppose $M \in_\alpha N \in_\alpha P$ and either $N$ is countable or $P$ is Magidor. Then $M \in_\alpha P$.*

We also need a version of intersection between countable and Magidor virtual models.

**Definition**

*Suppose* $M \in \mathcal{C}$ *and* $N \in \mathcal{U}$. *Let* $\bar{N}$ *be the transitive collapse of* $N$ *and let* $\pi$ *be the collapsing map of* $N$. *Suppose* $\bar{N} \in M$. *Then we let* $M \wedge N = \pi^{-1}[\bar{N} \cap M]$.

Finally, we need to define what it means for a node to be active at a certain $\alpha \in E$. First, given a virtual model $M$, let $\kappa_M = \sup(M \cap \kappa)$.

**Definition**

*Suppose $M$ is a virtual model and $\alpha \in E$.*

1. *$M$ is **strongly active** at $\alpha$ if $\eta(M) \geq \alpha$ and $M \cap E \cap \alpha$ is unbounded in $E \cap \alpha$.*
2. *$M$ is **active** at $\alpha$ if $\eta(M) \geq \alpha$ and $\mathrm{Hull}(M, V_{\kappa_M}) \cap E \cap \alpha$ is unbounded in $E \cap \alpha$.*

Note that a Magidor model $M$ is active at $\alpha$ iff it is strongly active at $\alpha$. However, for countable models there is a difference.

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Given $\mathcal{M} \subseteq \mathcal{C} \cup \mathcal{U}$ and $\delta \in E$ we let

$$\mathcal{M}^\delta = \{M \upharpoonright \delta : M \in \mathcal{M} \text{ and } M \text{ is active at } \delta\}.$$

**Definition (Main forcing)**

*Suppose $\alpha \in E \cup \{\lambda\}$. We let $\mathcal{M} \in \mathbb{M}_\alpha$ if it is a finite subset of $\mathcal{V}_{\leq\alpha}$ and, for every $\delta \in E \cap (\alpha+1)$, $\mathcal{M}^\delta$ is an $\in_\delta$-chain closed under $\wedge$.*

*We let $\mathcal{N} \leq \mathcal{M}$ if, for every $M \in \mathcal{M}$, there is $N \in \mathcal{N}$ such that $N \upharpoonright \eta(M) = M$.*

Note: Closure under $\wedge$ means: if $M, N \in \mathcal{M}^\delta$, $\bar{N} \in M$ and $N \wedge M$ is active at $\delta$ then it is in $\mathcal{M}^\delta$.

Suppose $\alpha < \beta$ and $M \in \mathbb{M}_\beta$. We let $\mathcal{M} \restriction \alpha = \{M \restriction \alpha : M \in \mathcal{M}\}$.

**Fact**

*Suppose $\alpha < \beta$. The map $\mathcal{M} \mapsto \mathcal{M} \restriction \alpha$ is a projection. Hence $\mathbb{M}_\alpha$ is a complete suborder of $\mathbb{M}_\beta$.*

**Proposition**

*$\mathbb{M}_\lambda$ is $\mathcal{C}_{\mathrm{st}} \cup \mathcal{U}$-strongly proper, where $\mathcal{C}_{\mathrm{st}}$ is the collection of* ***standard*** *virtual models.*

**Theorem**

*Let $G$ be $V$-generic over $\mathbb{M}_\lambda$. Then in $V[G]$, $\kappa = \omega_2$ and $\lambda = \omega_3$, and $2^\omega = \omega_3$.*

**Theorem**

*In $V[G]$ we have $\mathrm{ISP}(\omega_2)$, and if $\lambda$ is also supercompact, then $\mathrm{ISP}(\omega_3)$ holds as well.*

Let $\mathcal{M}_G = \bigcup G$. We can show that, if $\mu > \lambda$ and $N \prec V_\mu$ is a Magidor model containing all the relevant parameters and $N \cap V_\lambda \in \mathcal{M}_G$, then $N[G]$ is an $\omega_1$-guessing model in $V[G]$. For this we need to show the quotient forcing $\mathbb{M}_\kappa / G \cap N$ below the condition $\{N \cap V_\lambda\}$ is strongly proper for a stationary collection of countable models.

However, $V[G]$ does not satisfy $\mathrm{FS}(\omega_2)$. In order to arrange this, we need to add **decorations** to our conditions. This introduces some technical complications, so we do not discuss it.

# Thank You!

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