Partition Relations for Linear Orders without the Axiom of Choice

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Set Theory, Model Theory & Applications
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Folklore ([933Si])

Assume the Axiom of Choice. Then $L \not\to (\omega^*, \omega)^2$ *for any linear order* L.

Theorem ([965Kr, Theorem 8] and [971E, Theorem 5])

Assume the Axiom of Choice. Then

$$L \not\to (4, \omega^* + \omega)^3 \text{ and}$$
$$L \not\to (4, \omega + \omega^*)^3 \text{ for any linear orders } L.$$

Theorem ([971E, Theorem 5])

Assume the Axiom of Choice. Then $L \not\to (5, \omega^* + \omega \vee \omega + \omega^*)^3$ *for all linear orders* L.

Question

Assume the Axiom of Choice. Is there a linear order L *with* $L \to (4, \omega + \omega^* \vee \omega^* + \omega)^3$*?*

Theorem ([976Pr])

The axiom of determinacy of games of reals $\mathrm{AD}_{\mathbb{R}}$ *implies that* $\omega \to (\omega)^{\omega}_{2}$.

Theorem ([977Ma, 5.1 Metatheorem])

It is consistent from an inaccessible cardinal that $\omega \to (\omega)^{\omega}_{2}$.

Theorem (Donald Martin, [003Ka, Theorem 18.12], [004JM, 990Ja, 981K])

The axiom of determinacy AD *implies that* $\omega_1 \to (\omega_1)^{\omega_1}_{2}$.

Observation

$\langle {}^{\alpha}2, <_{lex} \rangle \not\to (\omega^*, \omega)^3$ *for all ordinals* α*.*

Theorem ([981Bl])

For every continuous colouring χ *with* $\mathrm{dom}(\chi) = [{}^{\omega}2]^n$ *there is a perfect* $P \subset {}^{\omega}2$ *on which the value of* χ *at an n-tuple is decided by its splitting type.*

Folklore ([967My, 978Ta])

Every relation on the reals with the property of Baire is continuous on a perfect set.

Observation

$\langle {}^{\alpha}2, <_{lex} \rangle \not\to (\omega^*, \omega)^3$ *for all ordinals* α*.*

Theorem ([981Bl])

For every *Baire* *colouring* χ *with* $\mathrm{dom}(\chi) = [{}^{\omega}2]^n$ *there is a perfect* $P \subset {}^{\omega}2$ *on which the value of* χ *at an n-tuple is decided by its splitting type.*

Folklore ([967My, 978Ta])

Every relation on the reals with the property of Baire is continuous on a perfect set.

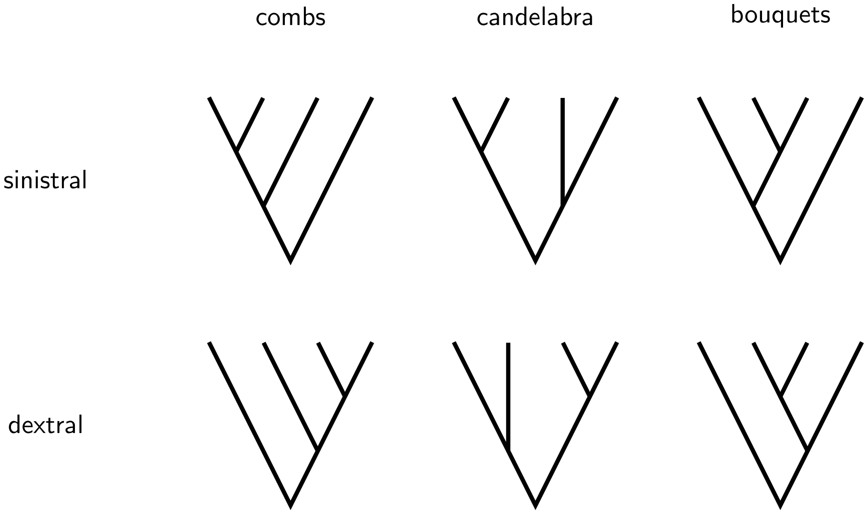


Figure: Combs, Candelabra and Bouquets

Theorem

Suppose that all sets of reals have the property of Baire.
Then $\langle {}^{\omega}2, <_{lex} \rangle \to (\langle {}^{\omega}2, <_{lex} \rangle)^2_n$ *for all* n.

Theorem

Suppose that all sets of reals have the property of Baire.
Then $\langle {}^{\omega}2, <_{lex} \rangle \to (\langle {}^{\omega}2, <_{lex} \rangle, 1 + \omega^* \vee \omega + 1)^3$.

Summary

Assume that all sets of reals have the property of Baire. Then

$$\langle {}^{\omega}2, <_{lex} \rangle \to (\omega + 1)^4_2,$$
$$\langle {}^{\omega}2, <_{lex} \rangle \to (5, 1 + \omega^* + \omega + 1 \vee \omega + 1 + \omega^*)^4,$$
$$\langle {}^{\omega}2, <_{lex} \rangle \to (6, 1 + \omega^* + \omega + 1 \vee m + \omega^* \vee \omega + n)^4 \text{ for all } m, n < \omega.$$

- $\vec{p}$ is a *cactus* if and only if s_p divides $\vec{p}$ in a comb of the same chirality and a branch.
- $\vec{p}$ is a *grape* if and only if s_p divides $\vec{p}$ in a comb of the opposite chirality and a branch.
- $\vec{p}$ is an *olivillo* if and only if s_p divides $\vec{p}$ in a bouquet of the same chirality and a branch.
- $\vec{p}$ is a *rose* if and only if s_p divides $\vec{p}$ in a bouquet of the opposite chirality and a branch.
- $\vec{p}$ is a *mistletoe* if and only if s_p divides $\vec{p}$ in a candelabrum and a branch.
- $\vec{p}$ is a *lilac* if and only if s_p divides $\vec{p}$ in a triple of the same chirality and a pair.
- $\vec{p}$ is a *guinea flower* if and only if s_p divides $\vec{p}$ in a a triple of the opposite chirality and a pair.

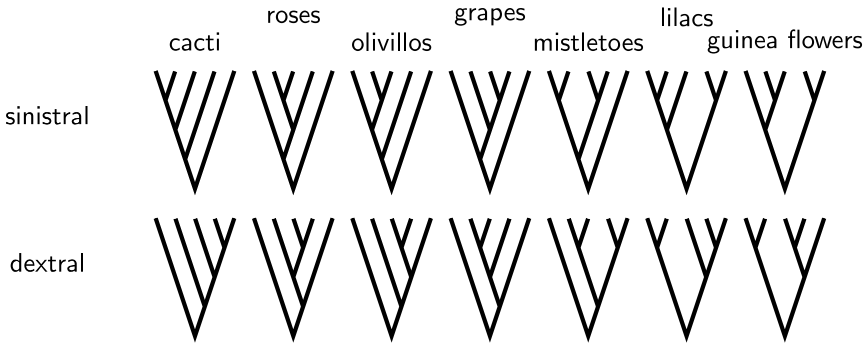


Figure: Seven (Fourteen) Pentapetalae, cf. [991HP, 009B&, 010M&, 015St]

Why 14? Catalan Numbers!

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900,$$

$$C_n := \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$$

- Number of triangulations of the $n+2$-gon,
- number of non-crossing partitions of $\{1, \ldots, n\}$,
- number of ways to insert n pairs of parentheses in a word of $n+1$ letters,
- et cetera.

Theorem

It is consistent that $\langle {}^{\omega_1}2, <_{lex}\rangle \to (\langle {}^{\omega_1}2, <_{lex}\rangle)^2$.

Theorem

Let κ *be an infinite initial ordinal and* $\alpha < \kappa^+$. *Then* $\langle {}^{\alpha}2, <_{lex}\rangle \not\to (2+\kappa^* \vee \omega, \omega^* \vee \kappa + 2)^m$ *for all* $m \geqslant 3$.

Summary

If α *is an ordinal, then the following statements hold.*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (5, \omega^* + \omega)^4,$$
$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (5, \omega + \omega^*)^4,$$
$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (7, \omega^* + \omega \vee \omega + \omega^*)^4.$$

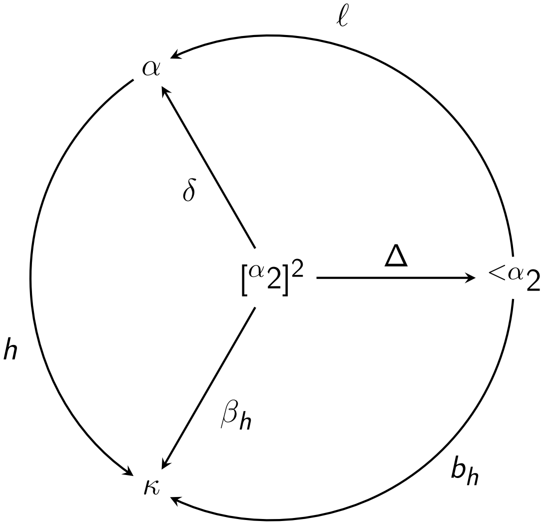


Figure: The functions $\Delta, \delta, \ell, h, b_h$ and β_h

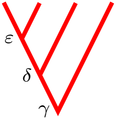
Lemma

For all ordinals α every sextuple within $\langle {}^{\alpha}2, <_{lex} \rangle$ contains a cactus, lilac, sinistral bouquet, dextral olivillo or dextral grape (and, by symmetry, a cactus, lilac, dextral bouquet, sinistral olivillo or sinistral grape).

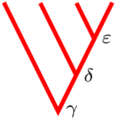
Lemma

Suppose that α is an infinite ordinal and $h : \alpha \hookrightarrow \#\alpha$ is an injection. For every $X \in [{}^{\alpha}2]^{\omega^*\omega}$, at least one of the following conditions hold.

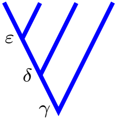
1. *There is a candelabrum $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4$ with $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))$.*
2. *There is a sinistral comb $\vec{x} = \{x_0, x_1, x_2, x_4\}_{<_{lex}} \in [X]^4$ with $\beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3)$.*



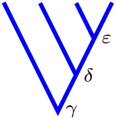
$b(\delta) < b(\varepsilon) < b(\gamma)$



$b(\varepsilon) < b(\gamma) < b(\delta)$



$b(\varepsilon) \notin b(\gamma) \setminus b(\delta)$



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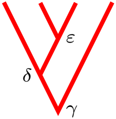
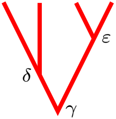
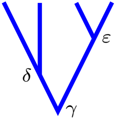


$b(\gamma) < b(\delta) < b(\varepsilon)$



$b(\gamma) < \min(b(\delta), b(\varepsilon))$



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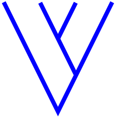
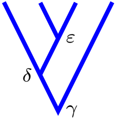
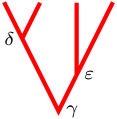




$b(\delta) \notin b(\varepsilon) \setminus b(\gamma)$



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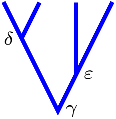

$b(\gamma) > \min(b(\delta), b(\varepsilon))$

Theorem

If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then

$$\begin{aligned}
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (2+\kappa^* \vee \kappa+2 \vee \eta, 5)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee (\kappa 2)^* \vee \kappa 2 \vee \kappa+2+\kappa^*, 5)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee \kappa+\omega \vee \omega^*+\kappa^*, 6)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\omega+\omega^* \vee 2+\kappa^* \vee \kappa+2, 6)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\kappa^*+\kappa \vee 2+\kappa^* \vee \kappa 2 \vee \omega\omega^*, 6)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee 2+\kappa^* \vee \kappa+\omega, 7)^4\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\kappa^*+\kappa \vee \kappa+2 \vee 2+\kappa^* \vee \eta, 7)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee \omega+\omega^* \vee (\kappa 2)^* \vee \kappa 2, 8)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\kappa^*+\omega \vee \omega^*+\kappa \vee 2+\kappa^* \vee \kappa+2 \vee \omega\omega^* \vee \omega^*\omega, 8)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee \omega+\omega^* \vee \kappa+2 \vee 2+\kappa^*, 9)^4.
\end{aligned}$$

Theorem

If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then

$$\begin{aligned}
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (2+\kappa^* \vee \kappa+2 \vee \eta, 5)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee (\kappa 2)^* \vee \kappa 2 \vee \kappa+2+\kappa^*, 5)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee \kappa+\omega \vee \omega^*+\kappa^*, 6)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\omega+\omega^* \vee 2+\kappa^* \vee \kappa+2, 6)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\kappa^*+\kappa \vee (\kappa 2)^* \vee \kappa+2 \vee \omega^*\omega, 6)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee \omega^*+\kappa^* \vee \kappa+2, 7)^4\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\kappa^*+\kappa \vee \kappa+2 \vee 2+\kappa^* \vee \eta, 7)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee \omega+\omega^* \vee (\kappa 2)^* \vee \kappa 2, 8)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\kappa^*+\omega \vee \omega^*+\kappa \vee 2+\kappa^* \vee \kappa+2 \vee \omega\omega^* \vee \omega^*\omega, 8)^4,\\
\langle {}^\alpha 2, <_{lex}\rangle &\not\to (\omega^*+\omega \vee \omega+\omega^* \vee \kappa+2 \vee 2+\kappa^*, 9)^4.
\end{aligned}$$

Question

Which partition relations of the form

$$\langle {}^{\kappa}2, <_{lex} \rangle \to \left(\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} L_\nu \right)^n$$

for $n \geqslant 3$ *are (jointly) consistent with* ZF $(+\,\mathsf{DC}_\kappa)$*, and which of the relations for* $\kappa = \omega_1$ *are provable in the theories* $\mathsf{ZF} + \mathsf{AD} + [V = L(\mathbb{R})]$ *and* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$*?*

Conjecture (W.)

Assume the Axiom of Determinacy. Then $\langle {}^{\omega_1}2, <_{lex} \rangle \to (6, \omega + \omega^* \vee \omega^* + \omega)^4$.

Proposition (Karagila, W., 2017)

If $\langle {}^{\omega_1}2, <_{lex} \rangle \to (6, \omega + \omega^* \vee \omega^* + \omega)^4$ *then the continuum cannot be well-ordered.*

Thank you!

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