Hereditarily Baire Hyperspaces

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Main result

All spaces are metrizable separable.

Theorem (Gartside-Medini-Z. 2017)

For a metrizable separable space X, $\mathcal{K}(X)$ is hereditarily Baire iff $\gamma X \setminus X$ has the Menger covering property for some (equivalently any) compactification γX of X. □

Recall that Y is *Baire* if every open $U \subset Y$ is non-meager. Y is *hereditarily Baire* (sometimes also called *completely Baire*) if every closed $Z \subset Y$ is Baire.
Polish spaces and compact spaces are hereditarily Baire.

For a filter on ω, hereditarily Baire is equivalent to being non-meager P (Marciszewski 1998).

Y is *Menger* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Remarks on the main theorem

If $\mathcal{K}(X)$ is hereditarily Baire, so is X because X is homeomorphic to a closed subspace of $\mathcal{K}(X)$.

The inverse implication fails: Let $B \subset \mathbb{R}$ be a Bernstein set, i.e., neither B nor $\mathbb{R} \setminus B$ has an uncountable compact subspace. Then $\mathcal{K}(B)$ is meager.

Let $\mathcal{F}$ be a Ramsey ultrafilter on ω. $\mathcal{F}$ is hereditarily Baire, but $\mathcal{K}(\mathcal{F})$ is not (because $\mathcal{F}$ is not Menger).

By a result of Kunen-Medini-Z., for every copy $Q \subset \mathcal{F}$ of the rationals there exists another such copy $Q' \subset Q$ such that $cl_{2^\omega}(Q') \subset \mathcal{F}$.

Motivation: Tukey order

Let P, Q be directed posets. Then $P \geq_T Q$ if there is a map $\phi : P \to Q$ that takes cofinal subsets of P to cofinal subsets of Q.

Equivalently (E. Schmidt 1955), $P \geq_T Q$ if there exists $\psi : Q \to P$ such that the image of each unbounded subset of Q is an unbounded subset of P.

Theorem (Christensen 1974)

If M is a separable metrizable space, then $\mathcal{K}(\omega^\omega) \geq_T \mathcal{K}(M)$ if and only if M is Polish.

Theorem (Gartside-Mamatelashvili 2016)

Let M be a separable metrizable space. Then $\mathcal{K}(M) \not\geq_T \mathcal{K}(\mathbb{Q})$ iff $\mathcal{K}(M)$ is hereditarily Baire.

Question

Is there a ZFC example of a space M such that $\mathcal{K}(M)$ is hereditarily Baire non-Polish? I.e., $\mathcal{K}(M) \not\leq_T \mathcal{K}(\omega^\omega)$ and $\mathcal{K}(\mathbb{Q}) \not\leq_T \mathcal{K}(M)$? Yes

Producing a ZFC example by citing :-)

Theorem (Hurewicz 1928)

X is hereditarily Baire iff it has no closed copy of $\mathbb{Q}$. □

We call X *σ-compactly controlled* if for every σ-compact $F \subset X$ there exists a Polish G such that $F \subset G \subset X$.

Observation

If X is X is σ-compactly controlled, then $\mathcal{K}(X)$ is hereditarily Baire. □

Theorem (Scheepers 1996)

X is σ-compactly controlled iff $Y := \gamma X \setminus X$ has the Hurewicz covering property. □

A topological space Y is *Hurewicz* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of Y there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of Y.

$\mathcal{U}$ is a γ-cover of Y if $\forall y \in Y \forall^* U \in \mathcal{U} (y \in U)$.

σ-compact $\rightarrow$ Hurewicz $\rightarrow$ Menger $\rightarrow$ Lindelöf.

Producing a ZFC example by citing :-)

Theorem (Scheepers 1996)

In ZFC there exists a Hurewicz non-σ-compact space $Y \subset 2^\omega$. $\square$

Putting all these facts together we get a ZFC example of a space X such that $\mathcal{K}(X)$ is hereditarily Baire non-Polish.

Producting a ZFC example directly

Observation

If Y is Menger, then $X := \gamma Y \setminus Y$ is hereditarily Baire, where γX is a metrizable compactification of X.

Proof.

If not, then by Hurewicz' theorem Y has a closed copy of ω^ω. But the Menger property is preserved by closed subspaces and ω^ω is not Menger. A contradiction. □

Witness for ω^ω being non-Menger:
$\mathcal{U}_n = \big\{\{x : x(n) = k\} : k \in \omega\big\}$.

Producting a ZFC example directly

Theorem (Gartside-Medini-Z. 2107)

For a metrizable separable space X, $\mathcal{K}(X)$ is hereditarily Baire iff $\gamma X \setminus X$ has the Menger covering property for some (equivalently any) compactification γX of X. $\square$

Proof.

$(\Leftarrow)$ Consider the map $\Phi : \gamma X \setminus X \to \mathcal{K}(\mathcal{K}(\gamma X))$, $y \mapsto \{K \in \mathcal{K}(\gamma X) : y \in K\}$. Easy to check that Φ is upper semicontinuous, and hence $\mathcal{Z} := \bigcup_{y \in \gamma X \setminus X} \Phi(y) \subset \mathcal{K}(\gamma X)$ is Menger. Note that $\mathcal{K}(X) = \mathcal{K}(\gamma X) \setminus \mathcal{Z}$ and use the Observation above. $\square$

Producing a ZFC example of a Menger non-σ-compact space

Given $x, y \in \omega^\omega$, $x \le^* y$ means $\{n : x(n) \le y(n)\}$ is cofinite.
$\mathfrak{d}$ is the minimal cardinality of a dominating subset of ω^ω.

Exercise: $|X| < \mathfrak{d} \to X$ is Menger.

A set $X \subset \omega^\omega$ is $\mathfrak{d}$-concentrated on a countable Q, if $|X| \ge \mathfrak{d}$ and $|X \setminus U| < \mathfrak{d}$ for any open U containing Q.

Observation

If Y is $\mathfrak{d}$-concentrated on $Q \subset Y$, then Y is Menger. □

Fact. There exists a $\mathfrak{d}$-concentrate set.

Proof. Fix a dominating $\{d_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ and inductively construct $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subset \omega^{\uparrow\omega}$ such that $s_\alpha \not\le^* d_\beta$ for all $\beta \le \alpha$. Viewed as a subspace of $(\omega+1)^{\uparrow\omega}$, S is $\mathfrak{d}$-concentrated on $Q = \{x \in (\omega+1)^{\uparrow\omega} : x$ is eventually $\omega\}$. □

Summarizing: For $X := (\omega+1)^\omega \setminus (S \cup Q)$, $\mathcal{K}(X)$ is hereditarily Baire non-Polish.

Remark. The first ZFC example of a Menger non-σ-compact $X \subset \mathbb{R}$ is due to Fremlin-Miller (1988), the construction given above appears first in a paper of Bartoszynski-Tsaban (2006).

Question (Fremlin 1991)

Is it consistent that there exists an analytic non-Borel $X \subset 2^\omega$ *such that* $\mathcal{K}(\mathbb{Q}) \geq_T \mathcal{K}(X)$*?*

Yes, by the following

Theorem (Gartside-Medini-Z. 2016)

(V=L). There exists an analytic non-Borel $X \subset 2^\omega$ *such that* $\mathcal{K}(\mathbb{Q}) >_T \mathcal{K}(X)$*.*

We construct a co-analytic Hurewicz $Y \subset 2^\omega$ such that $X = 2^\omega \setminus Y$ is as required. We use results of Vidnyanszky to make sure that Y is co-analytic, which extend and unify earlier results of A. Miller.

Adding more structure

Question (Banakh-Plichko 2006)

Is there a ZFC-example of a non-complete linear metric space X with hereditarily Baire hyperspace $\mathcal{K}(X)$?

Reformulation: Is there a ZFC-example of a non-complete linear metric space X with $Y := \gamma X \setminus X$ being Menger?

Theorem (Bella-Tokgoz-Z. 2015)

- *For a topological group X, $\gamma X \setminus X$ is Hurewicz iff X is Polish.*
- *Consistently, if $(\gamma X \setminus X)^2$ is Menger for a topological group X, then X is Polish.*

To prove the second one we modify Shelah's proof of Con(There are no P-points).

A seemingly related property

X is *countably controlled* if for every $Q \in [X]^\omega$ there exists a Polish G such that $Q \subset G \subset X$.

Observation

CH implies the existence of a countably controlled space X with $\mathcal{K}(X)$ being not hereditarily Baire. $\square$.

Question

Can X be equal to $\mathcal{F}^+$ for some filter $\mathcal{F}$? I.e., is it possible that $\mathcal{F}^+$ is countably controlled but $\mathcal{F}$ is non-Menger? What about $\mathcal{F} = \mathcal{F}(\mathcal{A})$ for some mad family $\mathcal{A}$ on ω? $\square$.

Recall that an infinite $\mathcal{A} \subset [\omega]^\omega$ is a mad family, if $A_0 \cap A_1$ is finite for any distinct $A_0, A_1 \in \mathcal{A}$, and $\mathcal{A}$ is maximal with respect to this property.

$\mathcal{A}$ is ω-mad if for every countable $\mathcal{B} \subset \mathcal{F}(\mathcal{A})^+$ there exists $A \in \mathcal{A}$ such that $|A \cap B| = \omega$ for all $B \in \mathcal{B}$.

Observation

If $\mathcal{A}$ is ω-mad, then $\mathcal{F}(\mathcal{A})^+$ is countably controlled. .

Proof.

Given a countable $\mathcal{B} \subset \mathcal{F}(\mathcal{A})^+$, pick a countable $\mathcal{A}_1 \subset \mathcal{A}$ such that $|A \cap B| = \omega$ for all $A \in \mathcal{A}_1$ and $B \in \mathcal{B}_1$.

$$G := \{Y \subset \omega : \forall A \in \mathcal{A}_1 \, (|A \cap Y| = \omega)\}$$

is as required. □

Proposition (Z. 2018)

In the Laver model, if $\mathcal{A}$ is ω-mad, then $\mathcal{F}(\mathcal{A})$ is Menger. □.

The last slide

Thank you for your attention.