On decidabilty of some classes of Stone algebras

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(Un)decidability results for Boolean algebras

The theory of the class **BA** of all Boolean algebras is decidable [A. Tarski 1949].

The theory of the class $\mathbf{BI}_n$ of Boolean algebras with a sequence of distinguished ideals $(B, J_1, \ldots, J_n)$ is decidable for each $n \geq 0$ [Ershov 1964], [Rabin 1969].

The theory of the class **BP** of Boolean pairs, i.e., Boolean algebras (B, S) with a distinguished subalgebra is hereditarily undecidable [Rubin 1976].

Stone algebras

Stone algebra $(A, \wedge, \vee, {}^*, 0, 1)$ is a distributive lattice with smallest element 0, biggest element 1, and unary pseudocomplement operation *, i.e.,

$$x \wedge y = 0 \;\Leftrightarrow\; y \leqslant x^*$$

satisfying

$$x^* \vee x^{**} = 1$$

They form an equational class [Grätzer 1967].

Skeleton, dense elements set and structural map

Skeleton and **dense elements set** of A, resp.:

$$\operatorname{Sk} A = \{a \in A \mid a \vee a^* = 1\}$$

$$\operatorname{Dn} A = \{d \in A \mid d^* = 0\}$$

$\operatorname{Sk} A$ is a Boolean subalgebra of A and $\operatorname{Dn} A$ is a filter in A.

Principal Stone algebra: if $\operatorname{Dn} A$ has the smallest element e.

Structural map: $(0, 1)$-lattice homomorphism

$h_A : (\operatorname{Sk} A, \wedge, \vee, 0, 1) \to (\operatorname{Dn} A, \wedge, \vee, e, 1), \;\; h_A(a) = a \vee e$

A is completely determined by $(\operatorname{Sk} A, \operatorname{Dn} A, h_A)$.

Triple construction

[Chen, Grätzer 1969], [Katriňák 1973]

Given Boolean algebra B, distributive lattice D (with 0 and 1) and homomorphism $h : (B, \wedge, \vee, 0, 1) \to (D, \wedge, \vee, 0, 1)$, we form the **P-product**

$$B \rtimes_h D = \{(b, d) \in B \times D \mid h(b) \geq d\}$$

$(0, 1)$ sublattice of $B \times D$;

$$(b, d)^* = \big(b^*, h(b^*)\big)$$

turns $B \rtimes_h D$ to a Stone algebra with

$$\mathrm{Sk}(B \rtimes_h D) = \big\{\big(b, h(b)\big) \mid b \in B\big\} \cong B$$

$$\mathrm{Dn}(B \rtimes_h D) = \big\{(1, d) \mid d \in D\big\} \cong D$$

and structural map $\widetilde{h} : \mathrm{Sk}(B \rtimes_h D) \to \mathrm{Dn}(B \rtimes_h D)$

$$\widetilde{h}\big(b, h(b)\big) = \big(b, h(b)\big) \vee (1, 0) = \big(1, h(b)\big)$$

Stone algebras of degree n

Iterating the triple construction with Boolean algebras we obtain the notion of **Stone algebra of degree** n.

One point Stone algebra is a Stone algebra of degree 0.

A Stone algebra is of degree $n + 1$ if its dense elements set forms a Stone algebra of degree n.

$\mathbf{SA}_n$ denotes the class of all Stone algebras of degree n.

$\mathbf{SA}_1$ is the class of all Boolean algebras and $\mathbf{SA}_n \subseteq \mathbf{SA}_{n+1}$ for each n.

Each $\mathbf{SA}_n$ is a finitely axiomatizable class.

$\mathrm{Th}\,\mathbf{SA}_0 = \{x = y\}$, $\mathrm{Th}\,\mathbf{SA}_1 = \mathrm{Th}\,\mathbf{BA}$,

$\mathrm{Th}\,\mathbf{SA}_{n+1} = \mathrm{Th}\,\mathbf{SA} \cup (\mathrm{Th}\,\mathbf{SA}_n)^{\mathrm{Dn}}$ for $n \geq 1$

The classes $\mathbf{SA}^{\mathrm{i}}_n$, $\mathbf{SA}^{\mathrm{s}}_n$

Iterated P-product [Katriňák, Mitschke 1972]

A Stone algebra A is in $\mathbf{SA}_n$ iff there is a finite sequence of Boolean algebras $B_1, B_2, \ldots, B_{n-1}, B_n$ and Boolean homomorphisms $h_k : B_k \to B_{k+1}$ $(1 \leqslant k < n)$, such that

$$A \cong B_1 \rtimes_{h_1} (B_2 \rtimes_{h_2} \ldots \rtimes_{h_{n-2}} (B_{n-1} \rtimes_{h_{n-1}} B_n) \ldots)$$
$$= \big\{(b_1, b_2, \ldots, b_{n-1}, b_n) \in B_1 \times B_2 \times \ldots \times B_{n-1} \times B_n \mid h_1(b_1) \geq b_2,\ h_2(b_2) \geq b_3, \ldots, h_{n-1}(b_{n-1}) \geq b_n\big\}$$

P-injective and **P-surjective** Stone algebras of order n, resp.

$\mathbf{SA}^{\mathrm{i}}_n$ — Stone algebras in $\mathbf{SA}_n$ with all the the h_k's injective

$\mathbf{SA}^{\mathrm{s}}_n$ — Stone algebras in $\mathbf{SA}_n$ with all the the h_k's surjective

The class $\mathbf{PA}_n$ of all Post algebras of degree n is definitionally equivalent to $\mathbf{SA}^{\mathrm{i}}_n \cap \mathbf{SA}^{\mathrm{s}}_n$.

Interpretations 1

Given two first order languages L, L', a p-ary **interpretation** $I : L \to L'$ consists of the following data [Rabin 1964]:

- an effectively computable mapping $x \mapsto x^I = \mathbf{x}$, assigning to each L-variable x a p-tuple of distinct L'-variables $\mathbf{x} = (x_1, \ldots, x_p)$, (for distinct x, y the lists $\mathbf{x}$, $\mathbf{y}$ are disjoint)
- an L'-formula $U(\mathbf{x})$, representing the universe of the interpretation
- an L'-formula $E(\mathbf{x}, \mathbf{y})$, representing the equality relation
- an L'-formula $R^I(\mathbf{x}^1, \ldots, \mathbf{x}^n)$ for each n-ary relational symbol R in L
- an L'-formula $f^I(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^n)$ for each n-ary functional symbol f in L (constants are nullary operation symbols)

Interpretations 2

I can be extended to a map $I : \text{Form}\, L \to \text{Form}\, L'$, $\varphi \mapsto \varphi^I$, by recursion:

$$\begin{array}{rcl}
(x = y)^I & \text{is} & E(\mathbf{x}, \mathbf{y}) \\
R(x^1, \ldots, x^n)^I & \text{is} & R^I(\mathbf{x}^1, \ldots, \mathbf{x}^n) \\
\big(x^0 = f(x^1, \ldots, x^n)\big)^I & \text{is} & f^I(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^n) \\
(\neg\, \varphi)^I & \text{is} & \neg\, \varphi^I, \\
(\varphi \star \psi)^I & \text{is} & \varphi^I \star \psi^I \text{ (for any binary connective } \star) \\
(\exists\, x\, \varphi)^I & \text{is} & (\exists\, x_1, \ldots, x_p)(U(\mathbf{x}) \;\&\; \varphi^I) \\
(\forall\, x\, \varphi)^I & \text{is} & (\forall\, x_1, \ldots, x_p)(U(\mathbf{x}) \Rightarrow \varphi^I)
\end{array}$$

Interpretations 3

An L'-structure $\mathfrak{M}$ with base set M **admits** I if

- $U_{\mathfrak{M}} = \big\{ \mathbf{a} \in M^p \mid \mathfrak{M} \models U(\mathbf{a}) \big\} \neq \emptyset$
- $E_{\mathfrak{M}} = \big\{ (\mathbf{a}, \mathbf{b}) \in U_{\mathfrak{M}} \times U_{\mathfrak{M}} \mid \mathfrak{M} \models E(\mathbf{a}, \mathbf{b}) \big\}$ is an equivalence relation on $U_{\mathfrak{M}}$
- $\mathfrak{M} \models \big(\forall\, x^1, \ldots, x^n\, \exists!\, x^0\;\, x^0 = f(x^1, \ldots, x^n) \big)^I$ for every n-ary functional symbol f in L
- $\mathfrak{M} \models \varepsilon^I$ for every instance ε of the equality axiom for any relational or functional symbol in L

Naturally defined L-structure $\mathfrak{M}^I$ with base set $U_{\mathfrak{M}}/E_{\mathfrak{M}}$ and relational and functional symbols interpreted as follows:

$$\begin{array}{lcl}
\mathfrak{M}^I \models R(\underline{\mathbf{a}}^1, \ldots, \underline{\mathbf{a}}^n) & \text{iff} & \mathfrak{M} \models R^I(\mathbf{a}^1, \ldots, \mathbf{a}^n) \\
\mathfrak{M}^I \models \underline{\mathbf{a}}^0 = f(\underline{\mathbf{a}}^1, \ldots, \underline{\mathbf{a}}^n) & \text{iff} & \mathfrak{M} \models f^I(\mathbf{a}^0, \mathbf{a}^1, \ldots, \mathbf{a}^n)
\end{array}$$

$\underline{\mathbf{a}}$ denotes the equivalence class of $\mathbf{a} \in U_{\mathfrak{M}}$ w.r.t. $E_{\mathfrak{M}}$

Interpretations 4

By induction, for any L-formula $\varphi(x^1, \ldots, x^n)$, $\mathbf{a}^1, \ldots, \mathbf{a}^n \in U_{\mathfrak{M}}$,

$$\mathfrak{M}^I \models \varphi(\underline{\mathbf{a}}^1, \ldots, \underline{\mathbf{a}}^n) \quad \text{iff} \quad \mathfrak{M} \models \varphi^I(\mathbf{a}^1, \ldots, \mathbf{a}^n)$$

In particular, for any L-sentence φ,

$$\mathfrak{M}^I \models \varphi \quad \text{iff} \quad \mathfrak{M} \models \varphi^I$$

A class $\mathbf{K}$ of L-structures is **definable** in a class $\mathbf{K}'$ of L'-structures if there is an interpretation $I : L \to L'$ such that every structure $\mathfrak{A} \in \mathbf{K}$ is isomorphic to the structure $\mathfrak{M}^I$ for some $\mathfrak{M} \in \mathbf{K}'$ which admits I.

I defines a **semantic embedding** of $\mathbf{K}$ into $\mathbf{K}'$.

An L-theory T is **interpretable** in an L'-theory T' if the class $\operatorname{Mod} T$ is definable in the class $\operatorname{Mod} T'$.

Proving (un)decidability 1

Theorem 0. Let T be a theory in a first order language L with finitely many non-logical symbols which is interpretable in a theory T' in a recursive first order language L'. Then

- if T is finitely axiomatizable and T' is decidable, then also T is decidable;
- if T is hereditarily undecidable (in particular, if T is finitely axiomatizable and undecidable), then also T' is hereditarily undecidable.

In order to prove **decidability** of some class $\mathbf{K}$, find a semantical embedding of $\mathbf{K}$ into a decidable class $\mathbf{K}'$.

In order to prove **undecidability** of some class $\mathbf{K}'$, find a semantical embedding of a hereditarily undecidable class $\mathbf{K}$ into $\mathbf{K}'$.

Proving (un)decidability 2

$\mathbf{SA}^{\mathrm{i}}_1 = \mathbf{SA}^{\mathrm{s}}_1 = \mathbf{SA}_1 = \mathbf{BA}$

Theorem 1. [M. Adamčík, PZ, Algebra Universalis 2012]
Let $n \geq 2$. Then

- the class $\mathbf{SA}^{\mathrm{i}}_n$ of all P-injective Stone algebras of degree n is hereditarily undecidable
- the class $\mathbf{SA}^{\mathrm{s}}_n$ of all P-surjective Stone algebras of degree n is decidable

Undecidability of $\mathbf{SA}^{\mathrm{i}}_n$

Theorem 1A. The class $\mathbf{SA}^{\mathrm{i}}_n$ of all P-injective Stone algebras of any degree $n \geq 2$ is hereditarily undecidable.

Sketch of proof:

As $\mathbf{SA}^{\mathrm{i}}_n \subseteq \mathbf{SA}^{\mathrm{i}}_{n+1}$ for each n, it suffices to prove hereditary undecidability of $\mathbf{SA}^{\mathrm{i}}_2$.

Every Boolean pair (B, S) can be regarded as a triple $(S, B, \mathrm{id} : S \to B)$ and that way it gives rise to a Stone algebra $S \rtimes_{\mathrm{id}} B \in \mathbf{SA}^{\mathrm{i}}_2$.

Conversely, every $A \in \mathbf{SA}^{\mathrm{i}}_2$ can be obtained in this way from the Boolean pair (B, S), with

$$B = \mathrm{Dn}\, A, \qquad S = h_A[\mathrm{Sk}\, A] \cong \mathrm{Sk}\, A$$

This enables to define a semantic embedding of the hereditarily undecidable class $\mathbf{BP}$ into $\mathbf{SA}^{\mathrm{i}}_2$.

Decidability of $\mathbf{SA}^{\mathrm{s}}_n$

Theorem 1B. The class $\mathbf{SA}^{\mathrm{s}}_n$ of all P-surjective Stone algebras of any degree $n \geq 2$ is decidable.

Sketch of proof:

Every Boolean algebra with $n-1$ ideals $J_1 \subseteq J_2 \subseteq \ldots \subseteq J_{n-1}$ defines the P-product

$$B \rtimes_{p_1} B/J_1 \rtimes_{p_2} \ldots \rtimes_{p_{n-1}} B/J_{n-1} \in \mathbf{SA}^{\mathrm{s}}_n$$

with canonic projections $p_1 : B \to B/J_1,\ p_k : B/J_{k-1} \to B/J_k$.

Conversely, every $A \in \mathbf{SA}^{\mathrm{s}}_2$ can be obtained in this way from its P-product representation

$$B_1 \rtimes_{h_1} B_2 \rtimes_{h_2} \ldots \rtimes_{h_{n-1}} B_n$$

with $J_k = (h_k \circ \ldots \circ h_1)^{-1}(0)$ for $1 \leqslant k \leqslant n-1$.

This enables to define a semantic embedding of $\mathbf{SA}^{\mathrm{s}}_n$ into the finitely axiomatizable decidable class class $\mathbf{BI}_{n-1}$.

Some consequences 1

Corollary 1. The following classes of algebras are hereditarily undecidable:

- the class $\mathbf{SA}_n$ of all n-th degree Stone algebras for $n \geq 2$, in particular the class $\mathbf{SA}_2$ of all Stone algebras with Boolean dense set
- the class of all relatively pseudocomplemented Stone algebras and the class $\mathbf{SA}$ of all Stone algebras
- the class of all Gödel algebras, i.e., Heyting algebras satisfying the identity $(x \to y) \vee (y \to x) = 1$

The last two items follow from the observation that all the algebras in $\mathbf{SA}_2$ (hence in every $\mathbf{SA}_n$) are relatively pseudocomplemented and satisfy $(x \to y) \vee (y \to x) = 1$ [Katriňák, Mitschke 1972], [Balbes, Dwinger 1974].

Some consequences 2

Corollary 2. The following classes of algebras are decidable:

- the class $\mathbf{PA}_n$ all Post algebras of degree n [Ershov 1967]
- the class $\mathbf{SAD}_n$ of all Stone algebras of degree n which are dually pseudocomplemented and form a dual Stone algebra under the operation of dual pseudocomplement

Reason: $\mathbf{PA}_n \subseteq \mathbf{SAD}_n \subseteq \mathbf{SA}_n^{\mathrm{s}}$ "up to definability"

Corollary 3. For each n the class $\mathbf{SA}_n$ of all Stone algebras of degree n has decidable first order theory of its finite members.

Reason: Each class $\mathbf{SA}_n$ can be singled out from the variety of Gödel algebras by a finite set of identities involving $\wedge$, $\vee$, $*$ and definable constants $e_0, e_1, \ldots, e_n$ [Katriňák, Mitschke 1972]. This variety (though undecidable) still has decidable first order theory of its finite members [K. and P. Idziak 1988].

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