Minicourse on Infinitary Ramsey Theory
I. Topological Ramsey spaces and applications to ultrafilters

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Outline of Part I

1. Ramsey Theory on the Baire space

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   (b) Properties of $L(\mathbb{R})[\mathcal{U}]$

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4. Applications of topological Ramsey spaces
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   (b) Properties of $L(\mathbb{R})[\mathcal{U}]$
   (c) And more...

Pigeons

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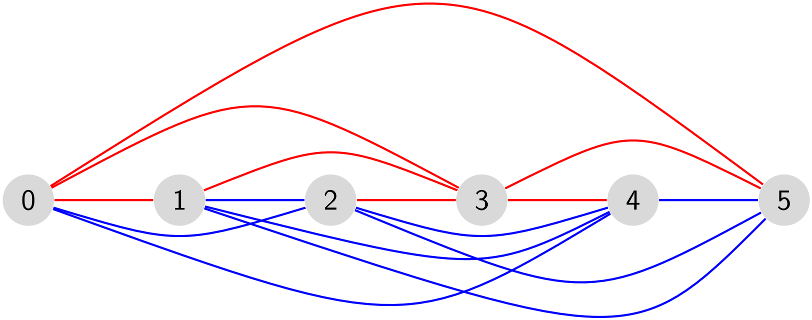
Pigeonhole Principle. Given a coloring $c : \omega \to 2$, there is an $i \in 2$ such that $c^{-1}(\{i\})$ is infinite.

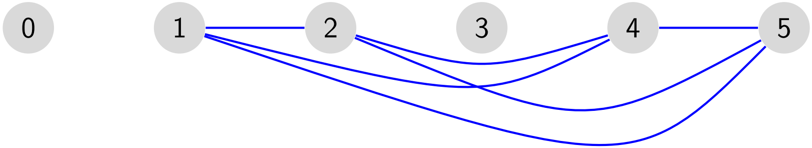
Ramsey's Theorem for Pairs of Natural Numbers

Given a coloring of pairs of natural numbers into red and blue:



Ramsey's Theorem for Pairs of Natural Numbers

There is an infinite subset M such that all pairs of numbers in M have the same color.



This can also be stated in terms of finding a complete infinite graph with all edges having the same color.

Ramsey's Theorem and Logic

Theorem. (Ramsey, 1929) Given $k, r \geq 1$ and a coloring

$$c : [\omega]^k \to r,$$

there is an infinite $N \subseteq \omega$ such that c takes only one color on $[N]^k$.

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This theorem appears in Ramsey's paper, *On a problem of formal logic*, and is motivated by Hilbert's Entscheidungsproblem:

Find a procedure for determining whether any given formula is valid.

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Ramsey applied his theorem to solve this problem for formulas with only universal quantifiers in front (Π_1).

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Write this as

$$\omega \rightarrow (\omega)^k_r$$

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Fact. (AC) There is a coloring $c : [\omega]^\omega \to 2$ so that for each $N \in [\omega]^\omega$, c takes both colors on $[N]^\omega$. That is,

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Thm. (Nash-Williams, 1965) Every clopen subset of $\mathcal{N}$ is Ramsey: Given a clopen subset $\mathcal{X} \subseteq \mathcal{N}$, for each $A \in [\omega]^\omega$ there is a $B \in [A]^\omega$ such that either $[B]^\omega \subseteq \mathcal{X}$ or $[B]^\omega \cap \mathcal{X} = \emptyset$.

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(ii) $\mathcal{F}$ is Nash-Williams: For $a, b \in \mathcal{F}$, $a \not\sqsubset b$.

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Thm. (Nash-Williams, 1965) Every front is Ramsey: that is, given a coloring c on a front $\mathcal{F}$ into 2 colors, there is an $M \in [\omega]^\omega$ such that c is monochromatic on $\mathcal{F}|M := \{a \in \mathcal{F} : a \subseteq M\}$.

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Pf Idea. Combinatorial Forcing

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The optimal extension of Ramsey's Theorem to infinite dimensions was given by Ellentuck. He introduced a topology refining the metric topology on the Baire space and showed that the Property of Baire characterizes those sets which are Ramsey.

Ellentuck Space $([\omega]^\omega, \subseteq, r)$

Basis for Ellentuck topology: $[a, X] = \{Y \in [\omega]^\omega : a \sqsubset Y \subseteq X\}$.

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The restriction map r will be discussed soon.

Connection with Mathias Forcing

Mathias forcing $\mathbb{M}$ has conditions (a, X), where $a \in [\omega]^{<\omega}$, $X \in [\omega]^{\omega}$, and $\max(a) < \min(X)$.

$(b, Y) \leq (a, X)$ iff $b \sqsupseteq a$, $Y \subseteq X$, and $b \setminus a \subseteq X$.

Mathias forcing is equivalent to forcing using the basic open sets in the Ellentuck space, partially ordered by $\subseteq$.

Connections with Forcing and Ultrafilters

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Ramsey ultrafilters have complete combinatorics. One way to state this is that if there is a supercompact cardinal in V, then any Ramsey ultrafilter in V is generic for the forcing $([\omega]^\omega, \subseteq^*)$ over the Solovay model $L(\mathbb{R})$.

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$\mathbb{M}$ is forcing equivalent to $([\omega]^\omega, \subseteq^*) * \mathbb{M}_{\mathcal{U}}$, where $\mathcal{U}$ is the Ramsey ultrafilter forced by $([\omega]^\omega, \subseteq^*)$.

Key properties from the Ellentuck space can be abstracted to give a general notion of a topological Ramsey space.

Abstract Topological Ramsey Spaces $(\mathcal{R}, \leq, r)$

$\mathcal{R}$ is a set, $\leq$ is quasi-order on $\mathcal{R}$.

For each n, $r_n(\cdot) := r(n, \cdot)$ is a restriction map on domain $\mathcal{R}$ giving the n-th approximation to X.

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Basic open sets: $[a, X] = \{Y \in \mathcal{R} : a \sqsubset Y \leq X\}$.

The topology on $\mathcal{R}$ generated by the basic open sets is a refinement of the 'metric topology' on $\prod_{n<\omega} \mathcal{AR}_n$.

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Def. (Todorcevic) A triple $(\mathcal{R}, \leq, r)$ is a topological Ramsey space if every subset of $\mathcal{R}$ with the Baire property is Ramsey, and if every meager subset of $\mathcal{R}$ is Ramsey null.

The Axioms **A.1** - **A.4**

A.1 (1) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

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For the Ellentuck space, $\mathcal{E} = [\omega]^\omega$, for $A = \{a_0, a_1, \dots\} \in \mathcal{E}$, enumerated in increasing order, $r_n(A) = \{a_i : i < n\}$. So **A.1** holds.

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A.2 There is a quasi-ordering $\le_{\text{fin}}$ on $\mathcal{AR}$ such that

(1) $\{a \in \mathcal{AR} : a \le_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,

(2) $A \le B$ iff $(\forall n)(\exists m)\ r_n(A) \le_{\text{fin}} r_m(B)$,

(3) $\forall a, b, c \in \mathcal{AR}[a \sqsubset b \wedge b \le_{\text{fin}} c \rightarrow \exists d \sqsubset c\ a \le_{\text{fin}} d]$.

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(3) $\forall a, b, c \in \mathcal{AR}[a \sqsubset b \wedge b \leq_{\mathrm{fin}} c \rightarrow \exists d \sqsubset c\ a \leq_{\mathrm{fin}} d]$.

$\mathcal{AE}_n = [\omega]^n$ and $\mathcal{AE} = [\omega]^{<\omega}$. For $a, b \in [\omega]^{<\omega}$, $a \leq_{\mathrm{fin}} b$ iff $a \subseteq b$ (and $\max(a) = \max(b)$). So **A.2** holds.

The Axioms **A.1** - **A.4**

$\mathrm{depth}_B(a)$ is the least n (if it exists) such that $a \leq_{\mathrm{fin}} r_n(B)$.
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A.3 (1): If $a \in [B]^{<\omega}$ and n is least such that $a \subseteq \{b_0, \dots, b_{n-1}\}$, then for each $A \in [r_n(B), B]$, $[a, A] \neq \emptyset$.

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(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\mathrm{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

A.3 (1): If $a \in [B]^{<\omega}$ and n is least such that $a \subseteq \{b_0, \dots, b_{n-1}\}$, then for each $A \in [r_n(B), B]$, $[a, A] \neq \emptyset$.

A.3 (2): If $A \in [B]^\omega$ and $a \subseteq A$, then there is an $A' \in [r_n(B), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

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For $a \in \mathcal{AR}$, $|a|$ denotes the k such that $a \in \mathcal{AR}_k$.

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If $k < \omega$, $a \in [B]^k$, $d = \mathrm{depth}_B(a)$, and $\mathcal{O} \subseteq [\omega]^{k+1}$, then there is an $A \in [B]^\omega$ with $r_d(B) \sqsubset B$ such that the set $\{c \in [A]^{k+1} : c \sqsupset a\}$ is either contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

Abstract Ellentuck Theorem

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Abstract Ellentuck Thm. (Todorcevic) If $(\mathcal{R}, \leq, r)$ satisfies **A.1 - A.4** and $\mathcal{R}$ is closed in $\mathcal{AR}^{\mathbb{N}}$ (with product topology), then $(\mathcal{R}, \leq, r)$ is a topological Ramsey space.

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The proof uses combinatorial forcing (recall proof of Nash-Williams Theorem).

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Classic Examples of Topological Ramsey Spaces

1. Ellentuck space
2. Carlson-Simpson space of equivalence relations on ω with infinitely many equivalence classes (dual Ramsey)
3. Pröml-Voigt spaces of parameter words and ascending parameter words
4. Milliken space of block sequences $\mathrm{FIN}_k^{[\infty]}$
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(D.-Mijares, 2015) has an example schema which encompasses (2) - (5) as special cases.

Ultrafilters forced by topological Ramsey spaces

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$\mathcal{G}$ usually induces an ultrafilter $\mathcal{U}$ on the countable base set $\mathcal{AR}_1$:

$$U \subseteq \mathcal{AR}_1 \text{ is in } \mathcal{U} \text{ iff } \exists X \in \mathcal{G}(\mathcal{AR}_1|X \subseteq U).$$

Selective Coideals and Complete Combinatorics

Given a topological Ramsey space $(\mathcal{R}, \le, r)$, a coideal $\mathcal{U} \subseteq \mathcal{R}$ is selective if for each $A \in \mathcal{U}$ and any collection $(A_a)_{a \in \mathcal{AR}|A}$ of members of $\mathcal{U} \restriction A$, there is a $U \in \mathcal{U}$ which diagonalizes $(A_a)_{a \in \mathcal{AR}|A}$.

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Thm. (Mijares, 2007) $(\mathcal{R}, \le)$ is forcing equivalent to $(\mathcal{R}, \le^*) * (\mathbb{M}^{\mathcal{R}}_{\dot{\mathcal{U}}}, \le)$, similarly to how Mathias forcing equivalent to $([\omega]^\omega, \subseteq^*) * \mathbb{M}_{\dot{\mathcal{U}}}$.

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Thm. (DiPrisco-Mijares-Nieto, 2017) In the presence of a supercompact cardinal, every selective coideal $\mathcal{U} \subseteq \mathcal{R}$ is generic for $(\mathcal{R}, \le^*)$.

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We now survey several of these inner topological Ramsey spaces.

Topological Ramsey spaces $\mathcal{R}_\alpha$, dense in Laflamme's forcings $\mathbb{P}_\alpha$, $\alpha < \omega_1$.

Ex. Laflamme's forcing to add a weakly Ramsey ultrafilter

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$Y \leq_1^* X$ iff $\forall^\infty n$, the n-th block of Y is contained in some block of X.

So $(\mathbb{P}_1, \leq_1^*)$ is like $([\omega]^\omega, \subseteq^*)$ except the partial ordering is more restrictive.

Thm. (Laflamme, 1989) $(\mathbb{P}_1, \leq_1^*)$ forces a weakly Ramsey ultrafilter.

$\mathcal{U}$ is weakly Ramsey if for each finitary coloring c of $[\omega]^2$, there is a $U \in \mathcal{U}$ for which c takes on at most two colors on $[U]^2$.

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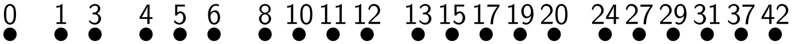
Complete Combinatorics: If κ is a Mahlo cardinal and G is Levy(κ)-generic over V, then any ultrafilter $\mathcal{U}$ on ω in $V[G]$ which is not Ramsey but is rapid and satisfies RP(k) for all k is generic over HOD$(\mathbb{R})^{V[G]}$

Laflamme's forcing $(\mathbb{P}_1, \leq_1)$. Example: $Y \leq_1 X$



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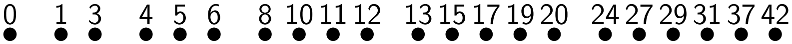

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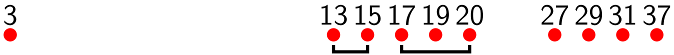

Figure: $Y \in [\omega]^\omega$ & $Y \leq_1 X$

The topological Ramsey space dense in $(\mathbb{P}_1, \leq_1)$

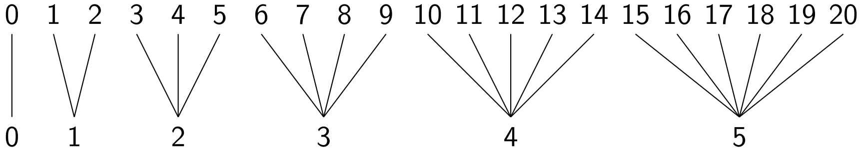

Figure: The maximum member of $\mathcal{R}_1$.

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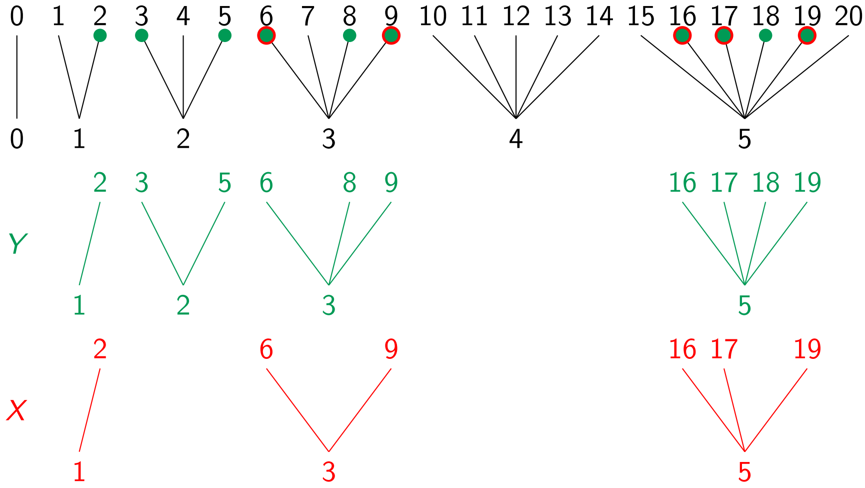

Figure: Two members X and Y of $\mathcal{R}_1$ with $X \leq Y$.

A subtree not in $\mathcal{R}_1$

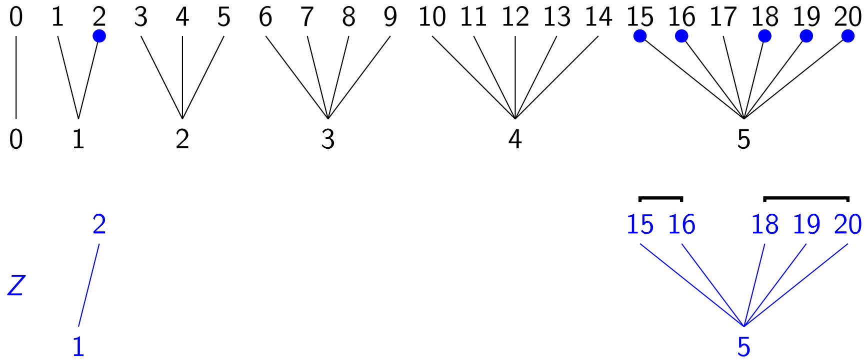

Figure: $Z \notin \mathcal{R}_1$

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Prop. For each $n \geq 2$, $\mathcal{U}_1 \to (\mathcal{U}_1)^n_{k,2^{n-1}}$.

This is stated in (Laflamme, 1989). An elegant proof using topological Ramsey space methods is given in Navarro Flores' Masters Thesis.

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(Laflamme, 1989) contains forcings $\mathbb{P}_\alpha$, $\alpha < \omega_1$, producing a hierachy of ultrafilters with weaker partition properties. Ramsey spaces $\mathcal{R}_\alpha$ dense inside these appear in (D.-Todorcevic, 2015).

Topological Ramsey spaces from Fraïssé classes.

These consolidate a forcing of Blass and
a collection of forcings by Baumgartner and Taylor.

Furthermore, they generate a class of new forcings producing
p-points with weak partition properties.

The n-square forcing of Blass

A subset of $\omega \times \omega$ of the form $s \times t$ is an n-square if $|s| = |t| = n$.

$X \subseteq \omega \times \omega$ is in $\mathbb{P}_{n-\mathrm{square}}$ iff for each $n < \omega$, X contains and $n \times n$-square.
For $X, Y \in \mathbb{P}_{n-\mathrm{square}}$, $Y \leq X$ iff $Y \subseteq X$.

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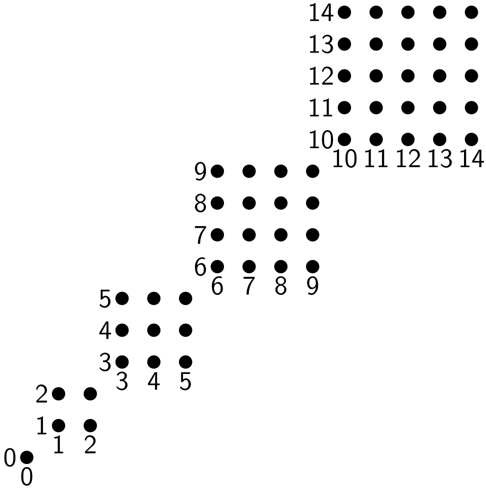
Thm. (Blass, 1973) $\mathbb{P}_{n-\mathrm{square}}$ forces a p-point with two Rudin-Keisler incomparable predecessors.

The Ramsey space $\mathcal{H}^2$ dense in the *n*-square forcing



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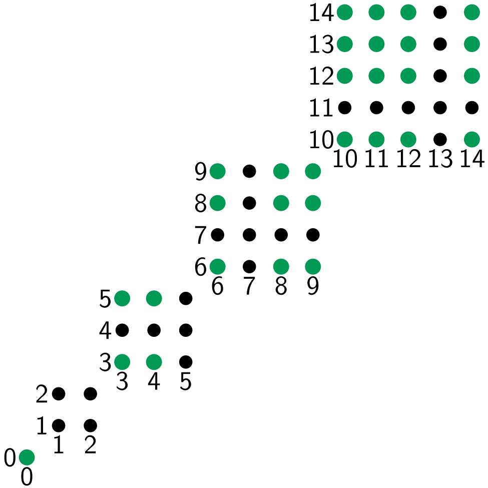

Figure: A member of X in $\mathcal{H}^2$.

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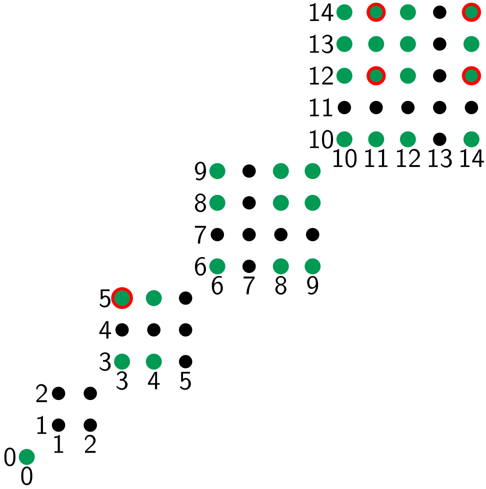

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The Hypercube topological Ramsey spaces

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Let $\mathcal{V}_2$ denote the ultrafilter forced by $(\mathcal{H}^2, \leq)$. Then $\mathcal{V}_2 \rightarrow (\mathcal{V}_2)^2_{k,6}$.

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Let $\mathcal{V}_2$ denote the ultrafilter forced by $(\mathcal{H}^2, \leq)$. Then $\mathcal{V}_2 \rightarrow (\mathcal{V}_2)^2_{k,6}$.

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Rem 2. Higher dimensional hypercube spaces were constructed in (D.-Mijares-Trujillo, 2017) including a space where the dimension of the n-th block is $n + 1$.

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Thm. (D.-Mijares-Trujillo, 2017) For each $k \geq 2$, there is a topological Ramsey space $\mathcal{A}_{k+1}$ which is dense in the Baumgartner-Taylor partial order forcing a k-arrow, not $(k+1)$-arrow ultrafilter.

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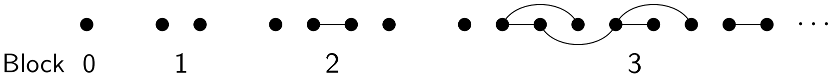

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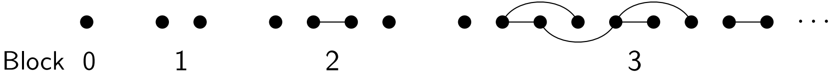

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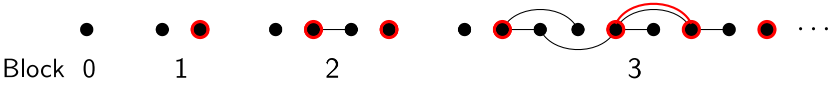

Figure: Example: Two members Y and X of $\mathcal{A}_3$ with $Y \leq X$

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(e.g. Masters Thesis work of Navarro Flores and new paper of (D.-Navarro Flores).

Motivation for Ramsey Spaces Dense in Forcings

$\mathcal{V} \leq_{RK} \mathcal{U} \Leftrightarrow$ there is a function $h : \omega \to \omega$ such that $\mathcal{V} = h(\mathcal{U})$,

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Def. The Fubini product of $\mathcal{U}$ and $\mathcal{V}$ is

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Let $\mathcal{K}_0, \dots, \mathcal{K}_n$ be any collection of Fraïssé classes of finite relational structures with the Ramsey property (and the OPFAP).

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3. If $\mathcal{V} \leq_T \mathcal{U}$, then the Tukey type $[\mathcal{V}]_T$ consists of all RK types of Fubini iterates of p-points essentially coded by members of $\mathcal{K}_0 \times \cdots \times \mathcal{K}_n$.

Exact Tukey and RK Structures: General Proof Outline

Let $\mathcal{R}$ be a topological Ramsey space, $\mathcal{U}$ be the filter forced by $(\mathcal{R}, \leq^*)$, and suppose $\mathcal{V} \leq_T \mathcal{U}$. Let $f : \mathcal{U} \to \mathcal{V}$ be a monotone cofinal map. Wlog assume $\mathcal{V}$ is an ultrafilter on base set ω.

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$$\mathcal{F}|X = \{a \in \mathcal{F} : \exists k < \omega \, (a \leq_{\text{fin}} r_k(X))\}.$$

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A general proof of (1) - (10) for a large class of spaces is given in (DMT).

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Note. E_I can be thought of as a projection map π_I, where $\pi_I(a) = \{a_i : i \in I\}$. Then $a \; \mathrm{E}_I \; b$ iff $\pi_I(a) = \pi_I(b)$.

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Next, extensions to Fraïssé classes.

Canonical Equivalence Relations for Fraïssé Classes

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Fronts and Barriers on $[\omega]^\omega$

$\mathcal{F} \subseteq [\omega]^{<\omega}$ is a front on $[\omega]^\omega$ iff

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Galvin's Lemma. Any front is a barrier, upon restriction to some infinite subset of ω.

Uniform fronts of rank $\alpha < \omega_1$

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Uniform fronts of higher rank are made recursively from those of lower rank.

Extension of Erdős-Rado Theorem to all fronts

(Can skip to slide 58 if pressed for time.)
For a front $\mathcal{F}$ and $M \in [\omega]^\omega$, $\mathcal{F}|M = \{a \in \mathcal{F} : a \subseteq M\}$.

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Pudlak-Rödl Canonization Thm. For every front (barrier) $\mathcal{F}$ on ω and every equivalence relation E on $\mathcal{F}$, there is an infinite $M \subseteq \omega$ such that $\mathrm{E} \restriction (\mathcal{F}|M)$ is represented by an irreducible map defined on $\mathcal{F}|M$.

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Exercise. The Pudlák-Rödl Theorem implies the Erdős-Rado Theorem.

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The finite rank fronts are of the form $\mathcal{AR}_k$ for some $k < \omega$. Recursively, one can extend the definition to infinite rank fronts.

The next family of topological Ramsey spaces forms the high dimensional analogues of the Ellentuck space.

They were found during the investigation of the initial structures of ultrafilters forced by $\mathcal{P}(\omega \times \omega)/\text{fin} \otimes \text{fin}$.

High Dimensional Ellentuck Spaces

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The projection to the first coordinates, $\pi_1(\mathcal{U}_2)$, is a Ramsey ultrafilter, generic for $\pi_1(\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}) \cong \mathcal{P}(\omega)/\mathrm{Fin}$.

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This construction of I_k can be extended to all uniform barriers on ω.

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Fact. If B projects to C, then $\mathcal{P}(C)/I_C$ embeds as a complete subalgebra of $\mathcal{P}(B)/I_B$, and $\mathcal{U}_C$ is isomorphic to a projection of $\mathcal{U}_B$.

Motivation for high dimensional Ellentuck spaces

Thm.

1. (Folklore) The ultrafilter $\mathcal{U}_2$ forced by $\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$ is Rudin-Keisler minimal above the Ramsey ultrafilter $\pi_1(\mathcal{U}_2)$.
2. (Blass-D.-Raghavan, 2015) $\mathcal{U}_2 \geq_T \pi_1(\mathcal{U}_2)$ and $\mathcal{U}_2$ is not Tukey maximal.

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3. (D. pre-preprint) For each uniform barrier B of infinite rank, $\mathcal{U}_B$ has initial Tukey and RK structures which are chains of length 2^ω, and they form a hierarchy via projection to barriers of smaller rank.

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A. Fix a particular order $\prec$ of the maximal nodes in this tree to get rid of any noise that could obstruct the pigeonhole property.

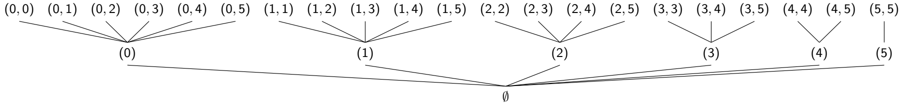
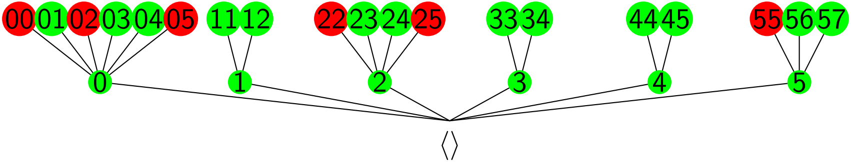
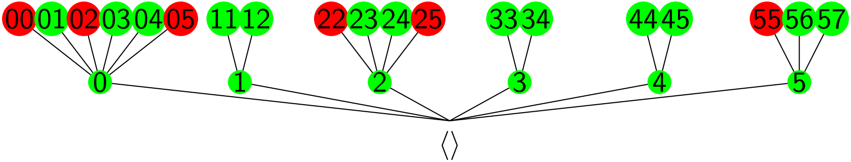


Figure: $\mathbb{W}_2$

$\mathcal{E}_2$ consists of all subsets of $\mathbb{W}_2$ for which the $\prec$-preserving bijection is also a tree-isomorphism.



A member of $\mathcal{E}_2$



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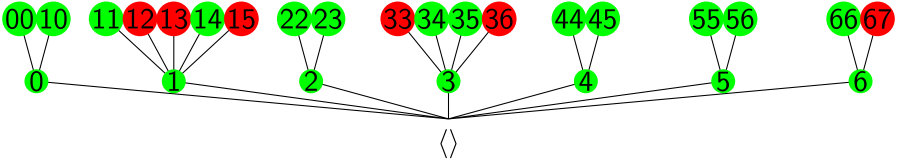

A member of $\mathcal{E}_2$

The collection of $X \subseteq \mathbb{W}_2$ for which the $\prec$-preserving bijection from $\mathbb{W}_2$ to X preserves the tree structure induces the finite approximations.
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Thm. (D., 2015) $\mathcal{E}_2$ satisfies Axioms **A.1** - **A.4**; hence is a topological Ramsey space.

The 3-dimensional Ellentuck space $\mathcal{E}_3$

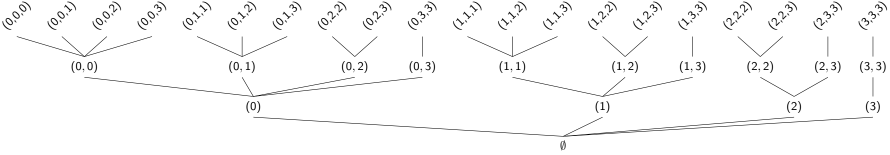
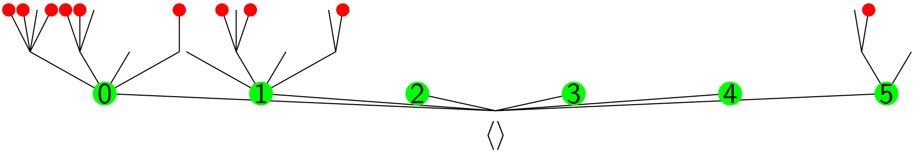


Figure: $\mathbb{W}_3$



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$\mathcal{E}_S$ for S the Schreier barrier

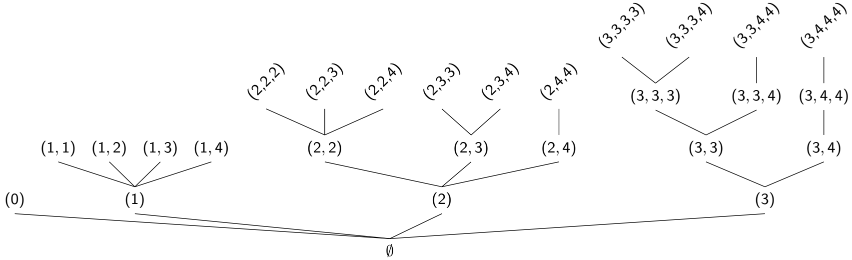

Figure: $\mathbb{W}_{\mathcal{S}}$

$X \in \mathcal{E}_S$ only if $X \subseteq \mathbb{W}_S$, for each n for which X has non-empty intersection with the subtree above (n), that restriction of X is in $\mathcal{E}_n$, and more structural requirements which are defined recursively from the structural requirements for th $\mathcal{E}_k$.

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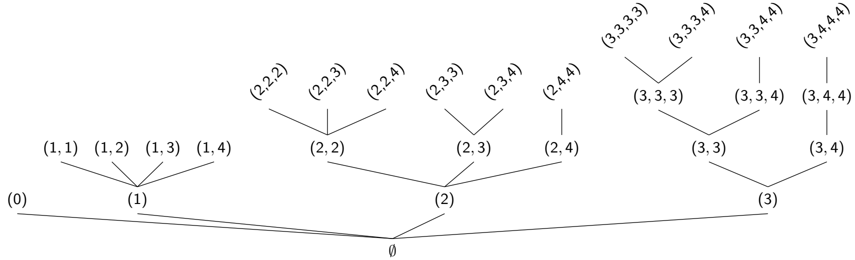

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The Infinite Dimensional Ellentuck Spaces

Thm. (D., 2016 JML) For each uniform barrier B, there is a topological Ramsey space $\mathcal{E}_B$ which is dense in I_B^+.

Hence, $(\mathcal{E}_B, \subseteq^{I_B})$ is forcing equivalent to $\mathcal{P}(B)/I_B$.

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Thus, the restriction of $\mathcal{P}(B)$ to $\mathcal{E}_B$ produces infinitary Ramsey theory, for those partitions into sets satisfying the property of Baire in the Ellentuck topology.

This was a necessary, though not sufficient, step in proving the initial Tukey structures below the ultrafilters $\mathcal{U}_B$.

Other Applications of Extended Ellentuck Spaces

A hierarchy of new Banach spaces

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My motivation for this project was to shed new light on distortion problems. Much work still needs to be done in this direction.

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She then showed that the high dimensional Ellentuck spaces satisfy the premises of this parametrization theorem, which is applied to obtain the theorem above.

Application: Barren Extensions

Thm. (Henle-Mathias-Woodin, 1985) Let M be a transitive model of ZF $+ \omega \to (\omega)^\omega$ and N its Hausdorff extension, that is the extension $M[\mathcal{U}]$ where $\mathcal{U}$ is the Ramsey ultrafilter forced by $\mathcal{P}(\omega)/\mathrm{Fin}$.

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In particular, this theorem holds when M is the Solovay model $L(\mathbb{R})$.

Classes of Barren Extensions

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In models of ZF in which all subsets of $\mathcal{R}$ are Ramsey, the AET implies that given $c : \mathcal{R} \to 2$, there is an $X \in \mathcal{R}$ such that c is monochromatic on $\{Y \in \mathcal{R} : Y \leq X\}$.

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This is central to the following result.

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This is central to the following result.

Thm. (D.-Hathaway) Let $\mathcal{R}$ be a topological Ramsey space satisfying a certain property $(*)$. Let M be a transitive model of ZF + every subset of $\mathcal{R}$ is Ramsey, and let $N = M[\mathcal{U}_\mathcal{R}]$ be the generic extension obtained by forcing with $(\mathcal{R}, \leq^*)$.
Then M and N have the same sets of ordinals; moreover, every sequence in N of elements of M lies in M.

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The property $(*)$ abstracts a property of the Ellentuck space. The new classes of topological Ramsey spaces discussed above all satisfy it.

This produces models $L(\mathbb{R})[\mathcal{U}]$ with stronger and stronger fragments of choice.

A good theorem to look into is Theorem 1.7 in *Introduction to Ramsey Spaces*, by Todorcevic.

It is a prelude to the Erdős-Rado Theorem mentioned above and has ideas which coincide with some material covered tomorrow.

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