Minicourse on Infinitary Ramsey Theory
II. Ramsey Theory on trees and applications to big Ramsey degrees

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University of Vienna, January 2019

Outline

1. Strong trees

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2. Halpern-Läuchli Theorem

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3. Milliken Theorem

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7. Open Problems

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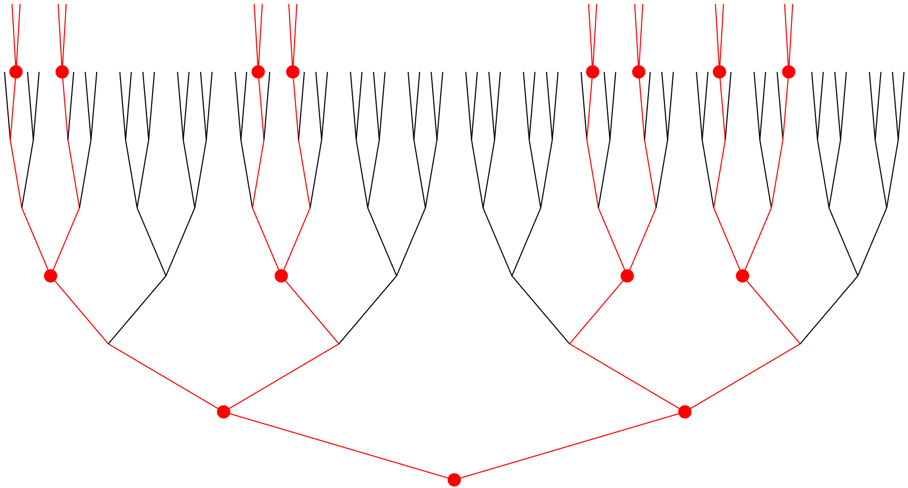
$S \subseteq T$ is a strong subtree of T iff there is an infinite set $\{m_n : n < \omega\}$ such that

1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < \omega$, $s \in S(n)$ and $u \in \mathrm{Succ}_T(s)$, there is exactly one $s' \in S(n+1)$ extending u.

Example: A Strong Subtree $S \subseteq 2^{<\omega}$



The nodes in S are of lengths $0, 1, 3, 6, \ldots$

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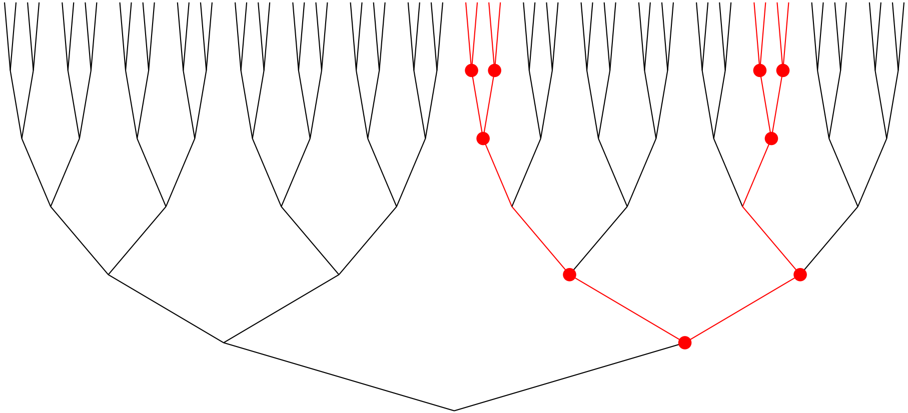

The nodes in U are of lengths $1, 2, 4, 5, \ldots$.

Halpern-Läuchli Theorem - strong tree version

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Theorem. (Halpern-Läuchli, 1966) Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes and let $r \geq 2$. Given a coloring $c : \bigotimes_{i<d} T_i \to r$, there are strong subtrees $S_i \leq T_i$ with nodes of the same lengths such that c is constant on $\bigotimes_{i<d} S_i$.

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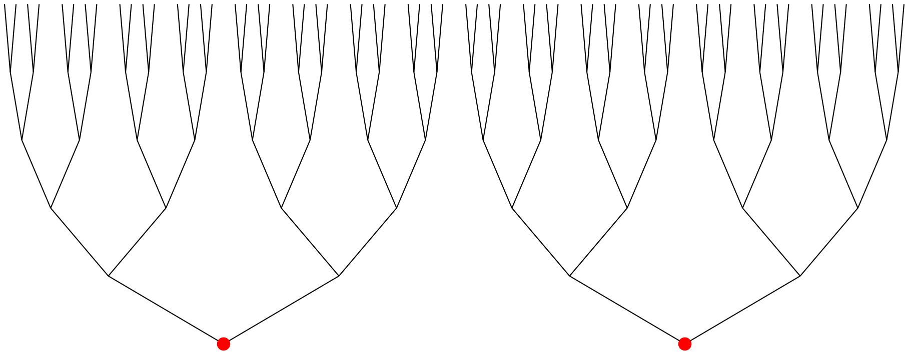
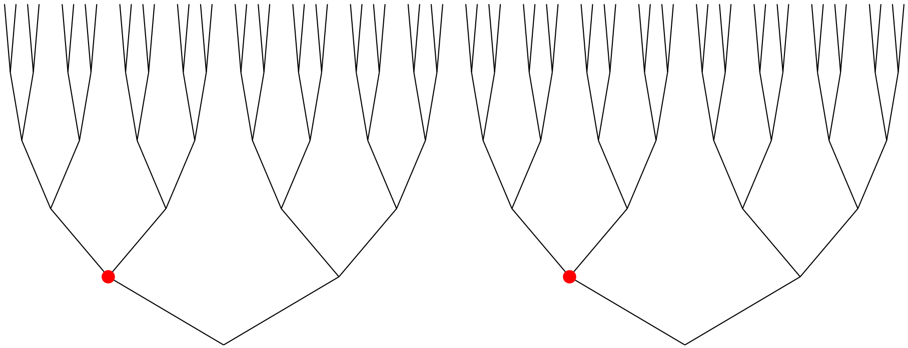
This was discovered as a key lemma in the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (See (Halpern-Lévy, 1971).)

We now give some examples of colorings of level products of two trees $T_0 = T_1 = 2^{<\omega}$, and show visually what the Halpern-Läuchli Theorem does.

Coloring Products of Level Sets: $T_0(0) \times T_1(0)$



Coloring Products of Level Sets: $T_0(1) \times T_1(1)$



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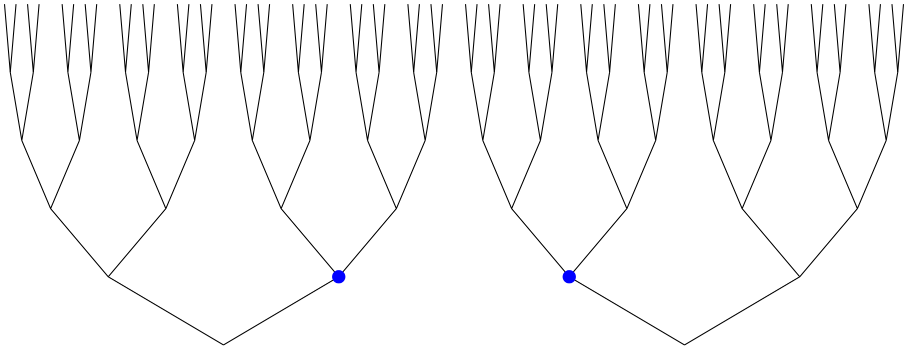

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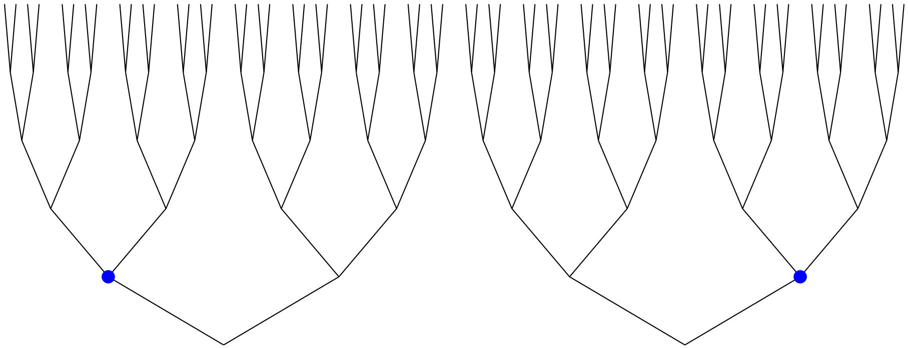

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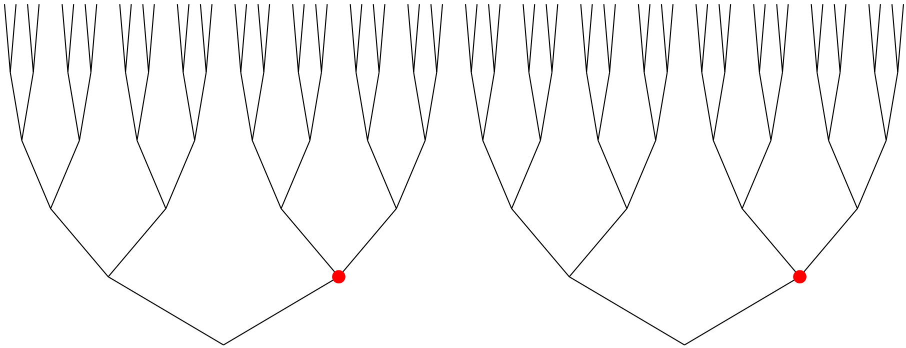
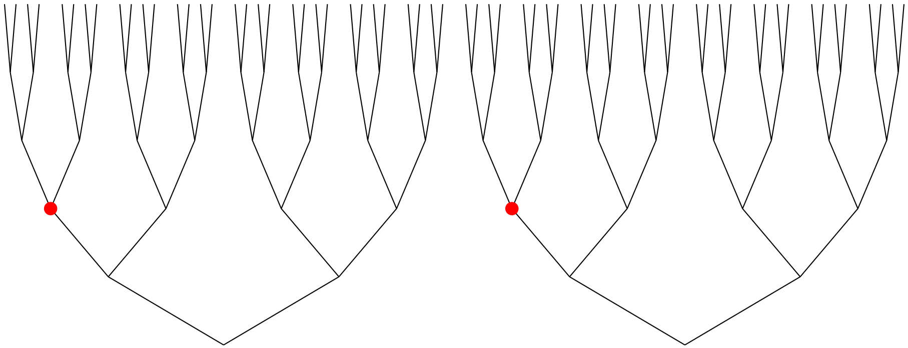


Coloring Products of Level Sets: $T_0(2) \times T_1(2)$



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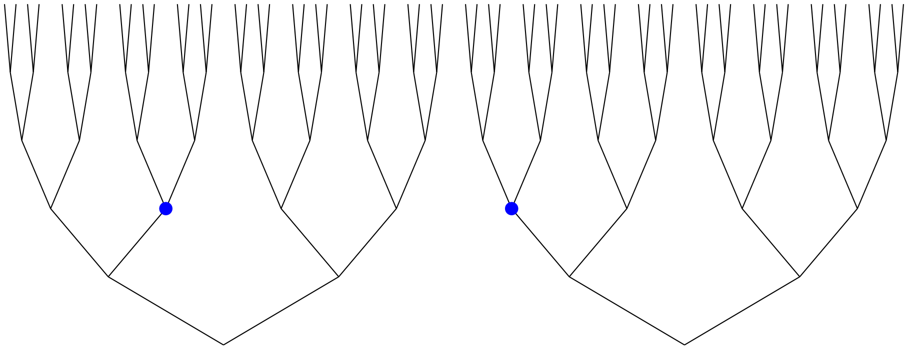

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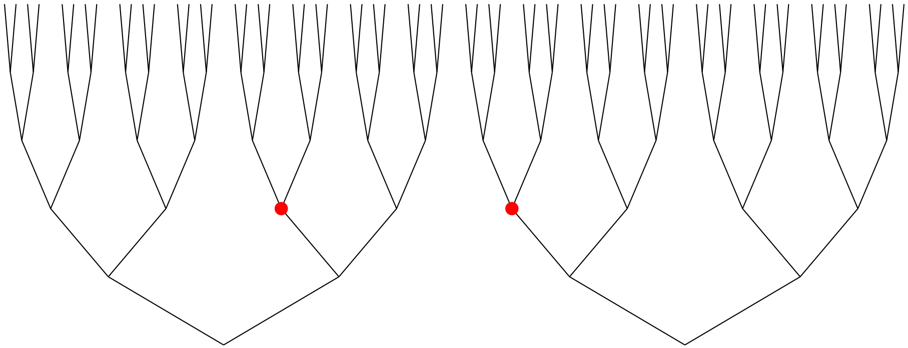

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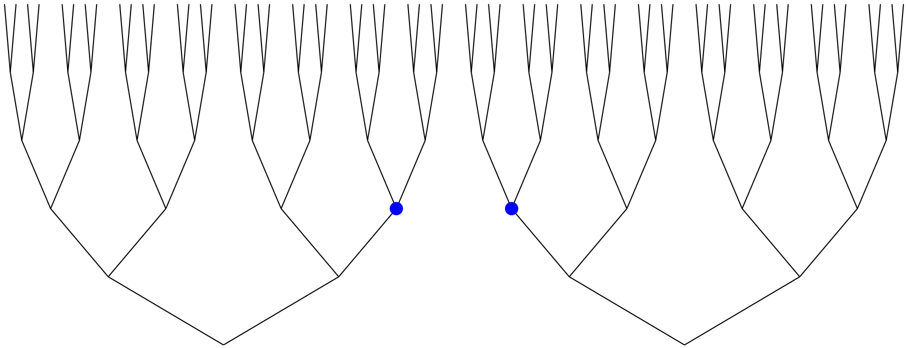

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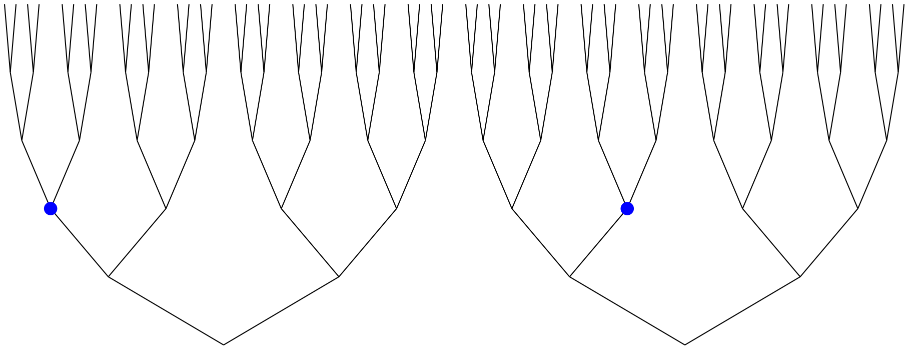

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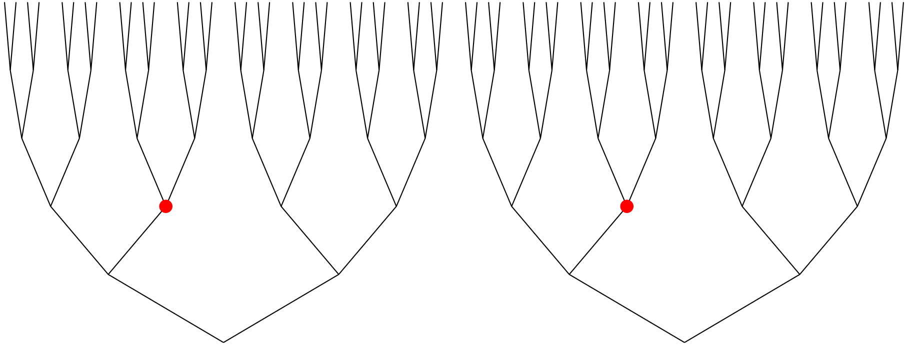
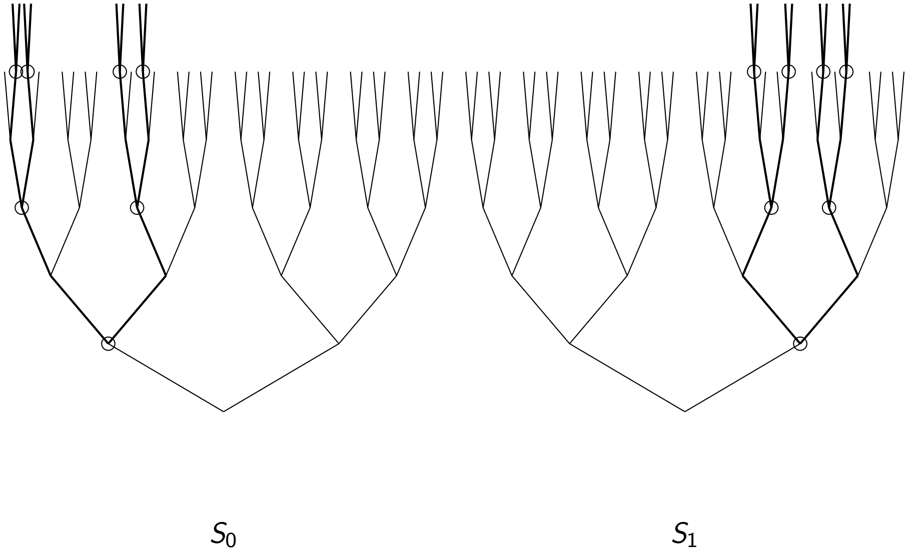


Etc.

HL gives Strong Subtrees with 1 color for level products



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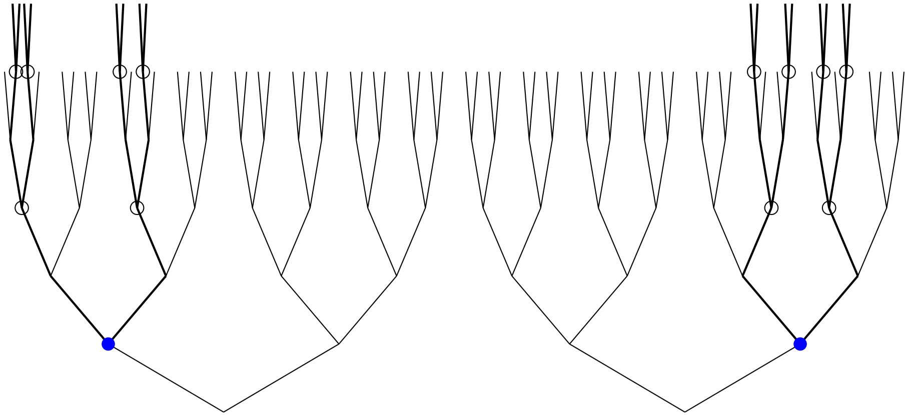

S_0 S_1

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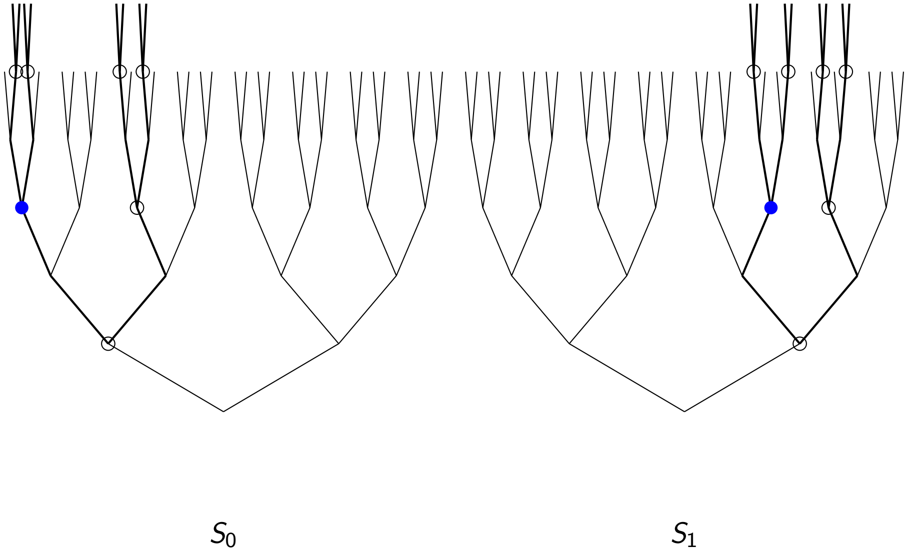

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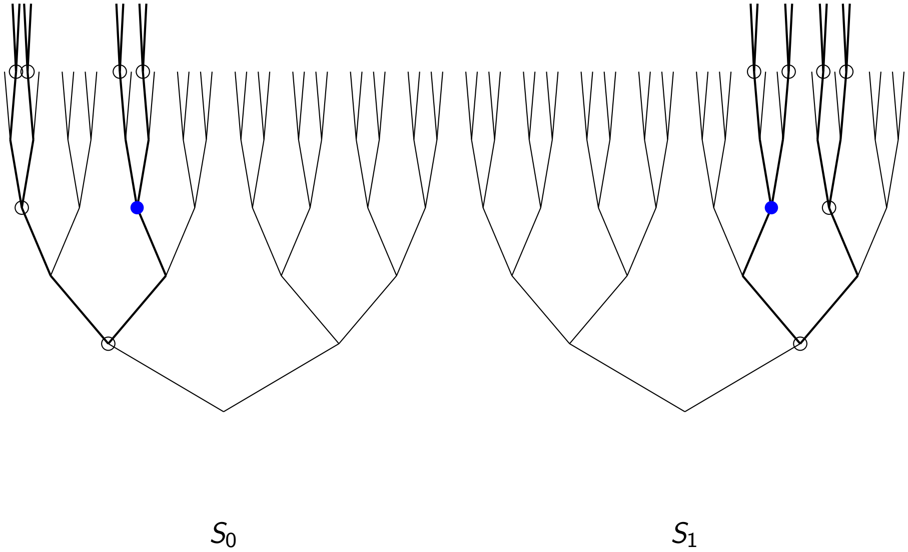

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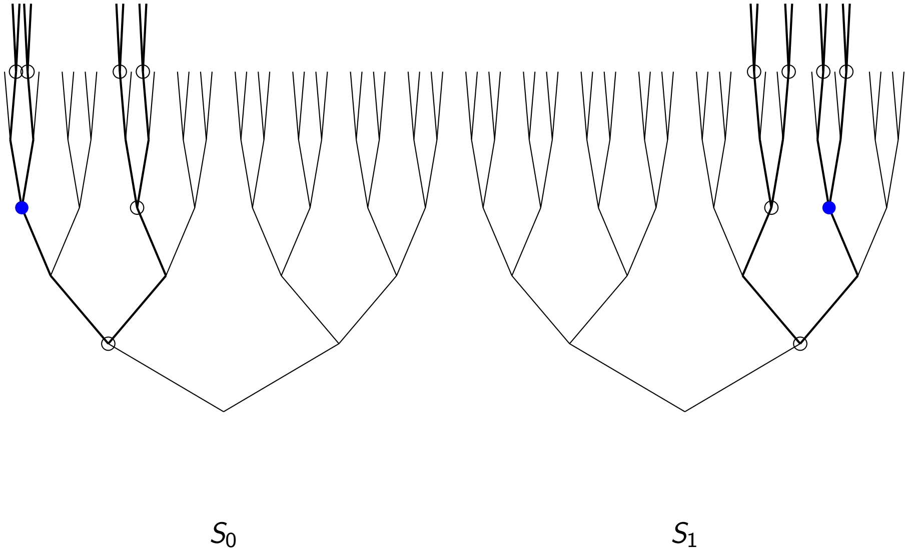

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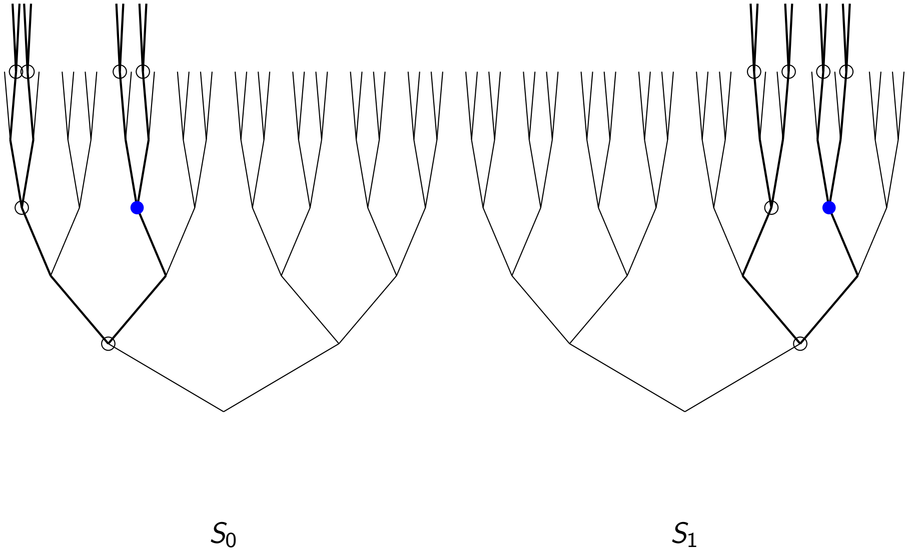

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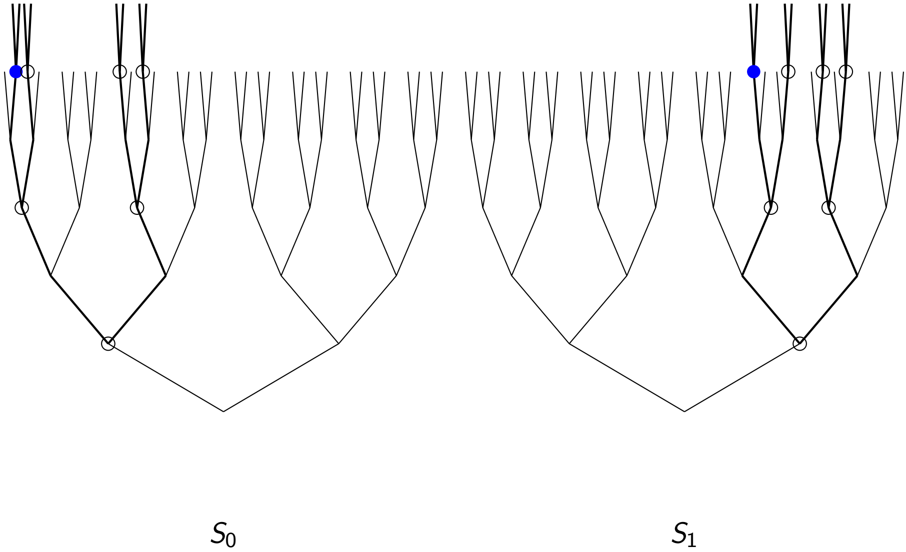

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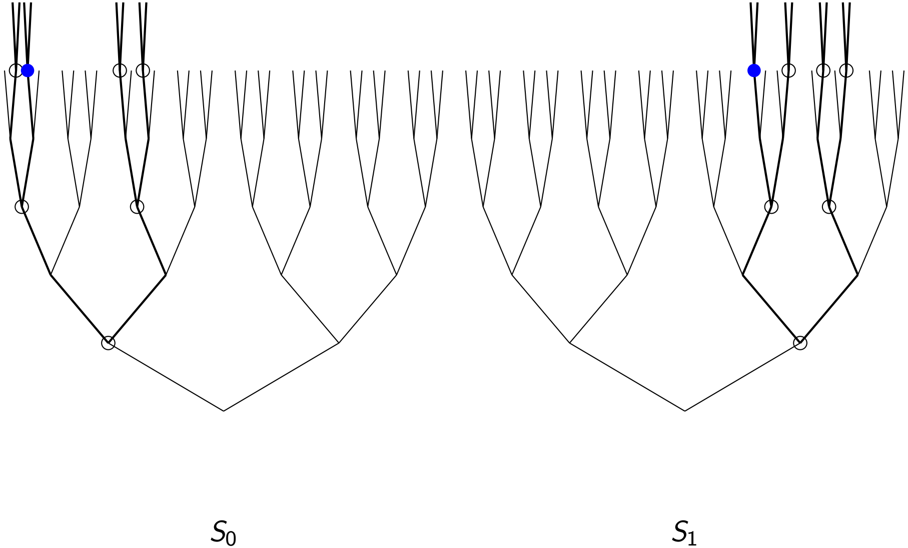

Application to Products of Rationals

Thm. (Laver, 1984) Given $d < \omega$ and a coloring of $\mathbb{Q}^d$ into finitely many colors, there are $X_i \subseteq \mathbb{Q}$, $i < d$, isomorphic to $\mathbb{Q}$ such that $X_0 \times \cdots \times X_{d-1}$ takes at most $d!$ many colors.

Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

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Let $\kappa = \beth_{2d}$, so that $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$.

Harrington's 'Forcing' Proof: The Forcing

The Forcing: $\mathbb{P}$ is the set of functions p of the form

$$p : d \times \vec{\delta}_p \to \bigcup_{i<d} T_i \upharpoonright l_p$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $l_p < \omega$, and $\forall i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \upharpoonright l_p$.

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$q \leq p$ iff $l_q \geq l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and $\forall (i,\delta) \in d \times \vec{\delta}_p$, $q(i,\delta) \supseteq p(i,\delta)$.

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$\mathbb{P}$ is a fancy form of Cohen forcing to add κ new reals.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i:

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For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1}\rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}\rangle$.

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- For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that
  1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
  2. $c(\{p_{\vec{\alpha}}(i,\alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

Harrington's 'Forcing' Proof: The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

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For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

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For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, &\langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ &\langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ &\langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle, \end{aligned} \tag{1}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order.

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where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}} \rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order. For $\vec{\theta} \in [\kappa]^{2d}$, define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$.

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Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

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Harrington's 'Forcing' Proof: Same ordinals, same position

Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

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$f(\iota, \vec{\mu}) = f(\iota, \vec{\nu}) = f(\iota, \vec{\theta})$ implies $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\beta}}(j') = \delta_{\vec{\alpha}}(j) = \delta_{\vec{\gamma}}(j')$,

which implies $j = j'$. $\square$

Harrington's 'Forcing' Proof: Set of compatible conditions

Lem 3. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

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By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i < d \rangle \in [k^*]^d$ such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$.

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Then for each $\vec{\alpha} \in \prod_{i<d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} =: t_i^*.$$

Harrington's 'Forcing' Proof: The Construction

Build strong subtrees $S_i \subseteq T_i$ homogeneous for c: Let $\text{stem}(S_i) = t_i^*$.

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Induction Assumption: $m \geq 1$, and we have constructed m-strong subtrees $\bigcup_{j<m} S_i(j)$ of T_i such that c takes color ε^* on $\bigcup_{j<m} \prod_{i<d} S_i(j)$.

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Let X_i be the set of immediate extensions in T_i of the nodes in $S_i(m-1)$. Let $J_i \subseteq [K_i]^{|X_i|}$. Label the nodes in X_i as $\{q(i,\delta) : \delta \in J_i\}$.

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Note that $q \leq p_{\vec{\alpha}}$, for all $\vec{\alpha} \in \vec{J}$.

Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

To construct $S_i(m)$, take $r \leq q$ for which $r \Vdash$ "$\forall \vec{\alpha} \in \vec{J},\ c(\dot{b}_{\vec{\alpha}} \restriction l_r) = \varepsilon^*$".

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Set $S_i = \bigcup_{m<\omega} S_i(m)$. c is monochromatic on $\bigotimes_{i<d} S_i$. $\square$ HL

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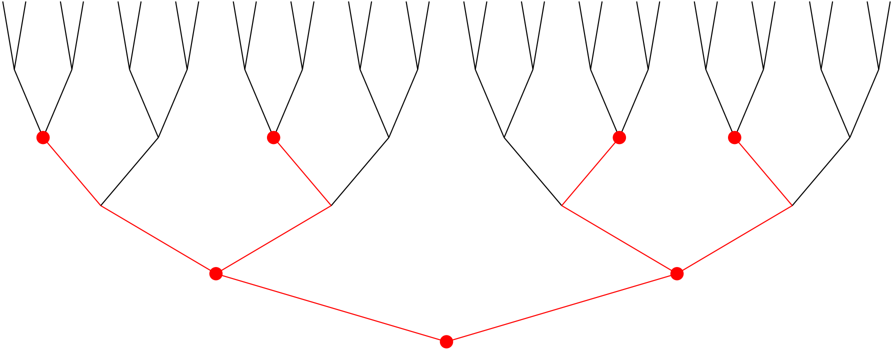
We give some examples for $T = 2^{<\omega}$.

Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

Given a coloring c of all 3-strong trees in $2^{<\omega}$ into red and blue:



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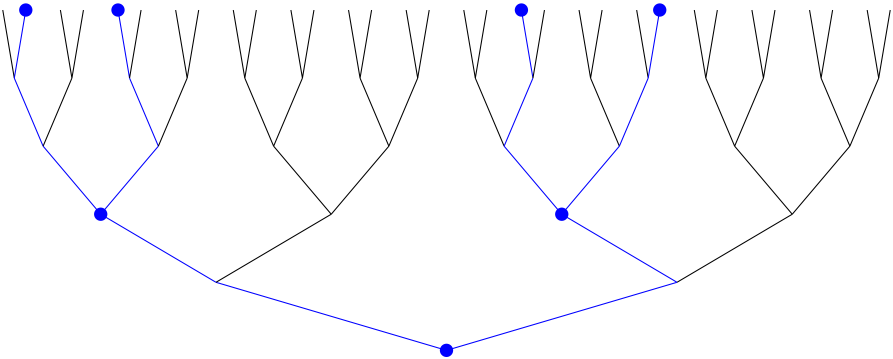

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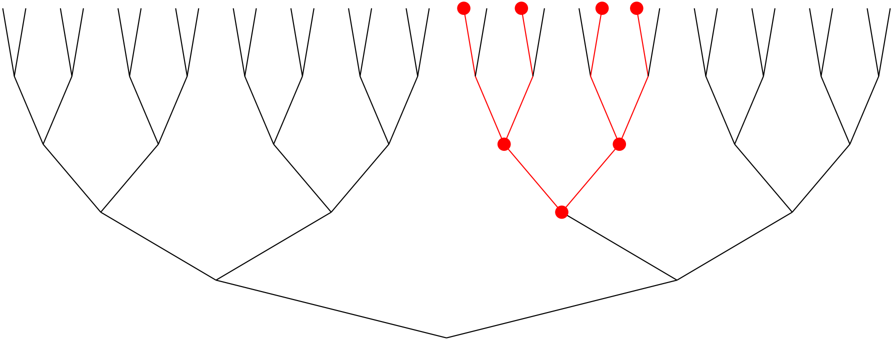

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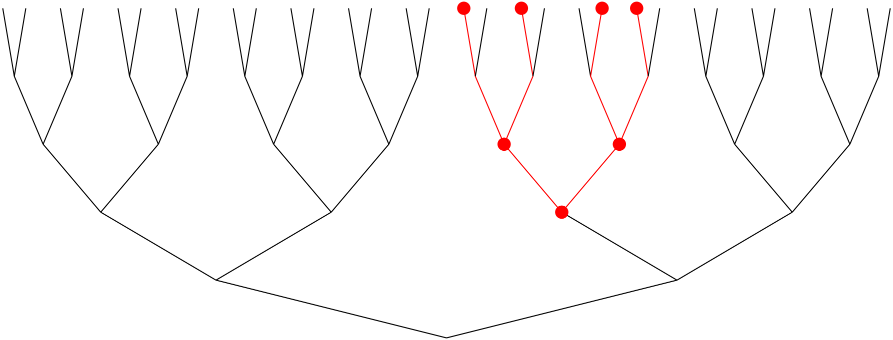

Milliken's Theorem guarantees a strong subtree in which all 3-strong subtrees have the same color.

Topological Ramsey Space of Strong Trees

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The Ellentuck topology is given by basic open sets

$$[A, T] = \{S \in \mathcal{S}_\infty : A \sqsubset S \subseteq T\},$$

where $\mathcal{S}_\infty$ denotes the set of all strong subtrees of T and A is a k-strong tree, for some $k < \omega$.

Infinite Structures with Analogues of Ramsey's Theorem

Infinite Ramsey's Theorem. Given $n, r \geq 1$ and a coloring $c : [\mathbb{N}]^n \to r$, there is an infinite subset $N \subseteq \mathbb{N}$ such that c is monochromatic on $[N]^n$.

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Here, we require the nice substructure to be isomorphic to the original infinite structure.

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• The exact numbers $T(n, \mathbb{Q})$ are tangent numbers! (Devlin 1979)

The Idea: Strong Tree Envelopes

$(\mathbb{Q}, <)$ can be coded by the nodes in $2^{<\omega}$.

For $s, t \in 2^{<\omega}$, define $s < t$ iff one of the following holds:

(a) $s \subsetneq t$ and $t(|s|) = 1$;

(b) $t \subsetneq s$ and $s(|t|) = 0$;

(c) $|s \wedge t| < \min(|s|, |t|)$, $s(|s \wedge t|) = 0$ and $t(|s \wedge t|) = 1$.

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Given a type, each finite tree in that tree is contained in finitely many strong tree envelopes. Apply Milliken's Theorem to these to obtain the upper bounds.

Big Ramsey Degrees of Infinite Structures

Where combinatorics, set theory, model theory, and topology meet.

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Question. Which infinite structures have finite big Ramsey degrees?

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• Actual degrees were found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).

Visuals: Colorings of Finite Graphs

Example: Ordered graph A embeds into ordered graph B.

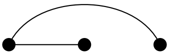

Figure: Ordered Graph A

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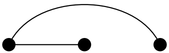


Figure: Ordered Graph A

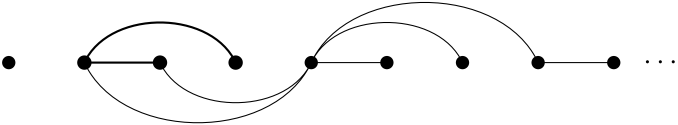
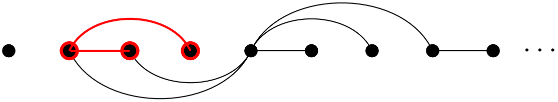


Figure: Ordered Graph B

Some copies of A in B



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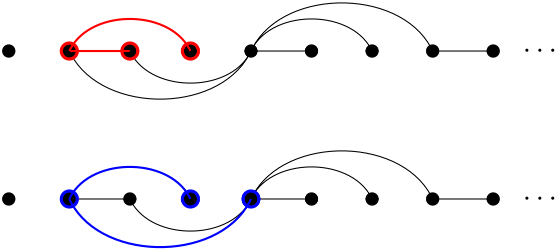

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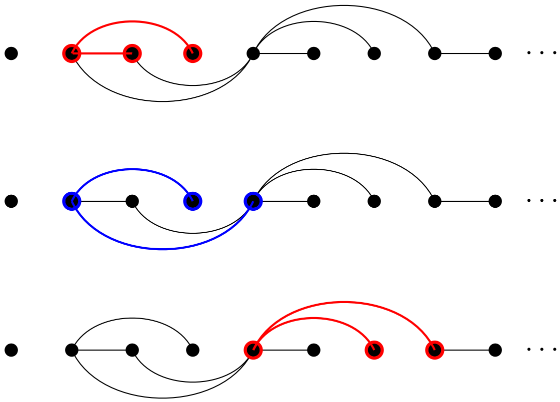

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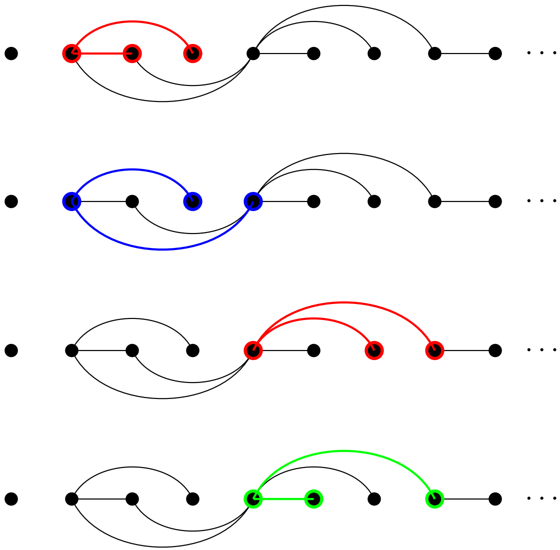

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The Fraïssé class of finite graphs has finite small Ramsey degrees.

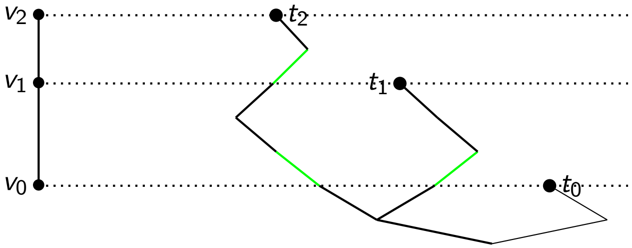
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \ E \ v_m \Leftrightarrow t_n(|t_m|) = 1.$$

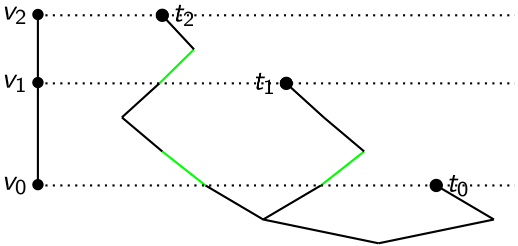
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Diagonal Trees Code Graphs

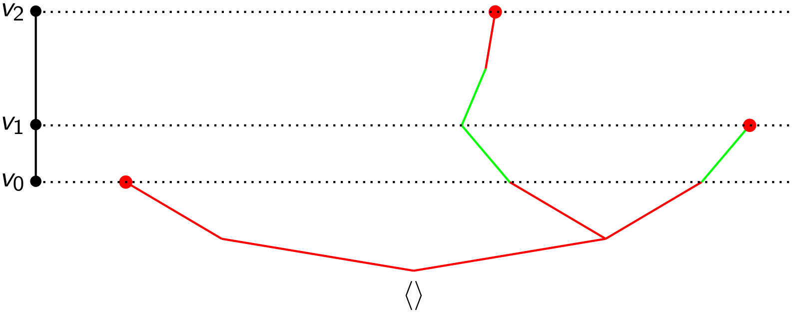
A tree T is diagonal if there is at most one meet or terminal node per level.

T is strongly diagonal if passing numbers at splitting levels are all 0 (except for the right extension of the splitting node).

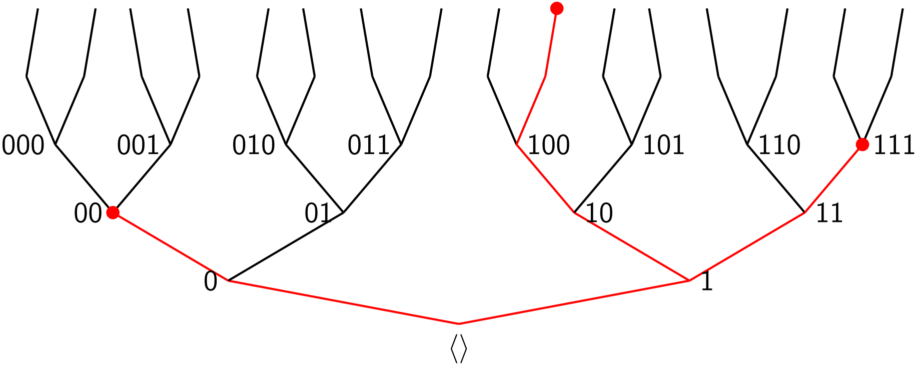


Every graph can be coded by the terminal nodes of a diagonal tree. Moreover, there is a strongly diagonal tree which codes $\mathcal{R}$.

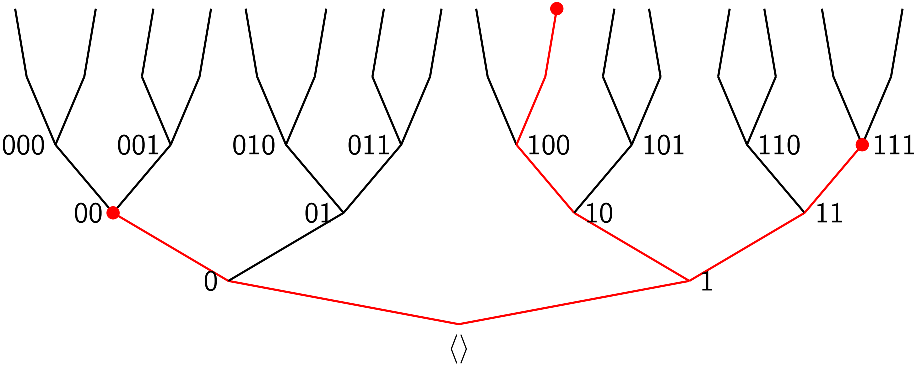
A Different Strongly Diagonal Tree Coding a Path



Strongly diagonal trees can be enveloped into strong trees



Another strong tree envelope



Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.

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Other Big Ramsey Structures

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Note. All of these have at their core either Ramsey's Theorem or an analogue of Milliken's Theorem.

Recent Work

The Henson graph $\mathcal{H}_k$ is the analogue of the Rado graph omitting all k-cliques, for $k \geq 3$.

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More on this at the Lecture.

Current work on several more classes, including $\mathbb{Q}_{\mathbb{Q}}$ from a question of Zucker at Banff 2018.

Connections with Topological Dynamics

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Recently, Zucker (2018) proved a correspondance for Fraïssé structures between finite big Ramsey degrees and universal completion flows.

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The first move towards this was on measurable cardinals.

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Thm. (Shelah 1991). Con(ZFC + $\exists\kappa$ measurable such that given $m < \omega$ and a coloring of $\bigcup_{\alpha<\kappa}[2^\alpha]^m$ into less than κ colors, there is a strong subtree $T \subseteq {}^{<\kappa}2$ on which the coloring takes only finitely many colors.)

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This left open questions about more than one tree.

$\mathrm{HL}(\delta, \sigma, \kappa)$

$T \subseteq \kappa^{<\kappa}$ is a regular tree if it is a κ-tree and all maximal branches have height κ.

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For regular trees $T_i \subseteq \kappa^{<\kappa}$, let

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For $\delta, \sigma > 0$ ordinals and κ an infinite cardinal, HL(δ, σ, κ) says:

Given a sequence $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ of regular trees and a coloring $c : \bigotimes_{i<\delta} T_i \to \sigma$, there exists a sequence of trees $\langle S_i : i < \delta \rangle$ such that

1. S_i is a strong subtree of T_i with the same set of levels $A \subseteq \kappa$ for each $i < \delta$, and
2. c is mononchromatic on $\bigotimes_{i<\delta} S_i$.

Somewhere Dense Version: SDHL(δ, σ, κ)

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For κ an infinite cardinal and ordinals $\delta, \sigma < \kappa$, $\mathrm{SDHL}(\delta, \sigma, \kappa)$ states:

Given regular trees $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ and a coloring

$$c : \bigotimes_{i<\delta} T_i \to \sigma,$$

there exist $\zeta < \zeta' < \kappa$, $\langle t_i \in T_i(\zeta) : i < \delta \rangle$, and $\langle X_i \subseteq T_i(\zeta') : i < \delta \rangle$ such that each X_i dominates $T_i(\zeta + 1) \cap \mathrm{Cone}(t_i)$ and

$$|c``\bigotimes_{i<\delta} X_i| = 1.$$

Questions Left Open from previous work

Question 1. For which uncountable cardinals κ and $\sigma < \kappa$ can $\mathrm{HL}(\delta, \sigma, \kappa)$ hold, either in ZFC or consistently?

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Question 3. For κ uncountable, given a fixed number of trees, are the various versions of the Halpern-Läuchli Theorem equivalent?

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Fact. (D./Hathaway) For every infinite cardinal κ and each positive integer k, $\mathrm{SDHL}(1, k, \kappa)$ holds.

This was improved by Zhang to $< \kappa$ colors and the asymmetric version.

Halpern-Läuchli at a measurable for several trees

Thm. (D./Hathaway) Let $d \geq 1$ be any finite integer and suppose κ is a $\kappa + d$-strong cardinal in a model V of ZFC satisfying GCH. Then there is a forcing extension in which κ remains measurable and $\mathrm{HL}(d, \sigma, \kappa)$ holds, for all $\sigma < \kappa$.

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Remark. This theorem lowers the natural upper bound on consistency, which would be $\kappa + 2d$-strong.

Conjecture. The consistency strength of $\mathrm{HL}(d, \sigma, \kappa)$ for κ measurable is a $\kappa + d$-strong cardinal.

Thm. (D/H) Let $\delta < \kappa < \lambda$ with κ, λ cardinals for which the following partition relations hold:

1) $\lambda \to (\kappa)^{\delta}_{\kappa}$, and
2) $\kappa \to (\mu_1)^{\delta \cdot 2}_{\mu_2}$, for all pairs $\mu_1, \mu_2 < \kappa$.

Assume κ is a measurable cardinal in the forcing extension to add λ many Cohen subsets of κ. Then $\mathrm{HL}(\delta, \sigma, \kappa)$ holds.

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In particular, we prove the stronger asymmetric version.

The proof uses ideas of Todorcevic's presentation of the Harrington argument for countable trees, but makes use of the measurable cardinal where the induction argument for countable trees fails for uncountable height trees.

Tail cone version and anaologue of Laver's Theorem

For $1 \leq d < \omega$, $\mathrm{HL}^{tc}(d, < \kappa, \kappa)$ says:

Given regular κ-trees $\langle T_i : i < d \rangle$, cardinals $\langle \sigma_\zeta < \kappa : \zeta < \kappa \rangle$, and colorings $c_\zeta : \bigotimes_{i<d} T_i \to \sigma_\zeta$ for $\zeta < \kappa$, there exist strong subtrees $\langle T_i' : i < d \rangle$ with the same set of levels $A = \{\alpha_\zeta : \zeta < \kappa\}$ such that for each pair $\zeta \leq \xi < \kappa$, given any $\langle t_i \in T_i'(\alpha_\xi) : i < d \rangle$, we have

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$$c_\zeta\langle t_i : i < d\rangle = c_\zeta\langle t_i \restriction \alpha_\zeta : i < d\rangle.$$

Thus, the c_ζ-color of a tuple $\vec{t} = \langle t_i : i < d\rangle$ on the ξ-th splitting level (for $\xi \geq \zeta$) is the same as the c_ζ-color of $\vec{t}$ restricted to the ζ-th splitting level.

Tail cone version and application

Thm. (Zhang) Let $d < \omega$ and $\kappa < \lambda$ be cardinals such that

1. $\lambda \to (\kappa)^{2d}_{2^\kappa}$; and
2. κ is inaccessible and is measurable in the forcing extension by $\mathrm{Add}(\kappa, \lambda)$.

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Thm. (Zhang) The analogue of Laver's theorem for the κ-rationals, κ measurable, is consistent.

Effects of forcing on HL(d, σ, κ)

Thm. (D/H) Let $0 < \sigma < \kappa$ with κ strongly inaccessible, and $0 < d < \omega$. Assume that $\mathrm{HL}(d, \sigma \cdot |\mathbb{P}|, \kappa)$ holds.

Effects of forcing on $\mathrm{HL}(d, \sigma, \kappa)$

Thm. (D/H) Let $0 < \sigma < \kappa$ with κ strongly inaccessible, and $0 < d < \omega$. Assume that $\mathrm{HL}(d, \sigma \cdot |\mathbb{P}|, \kappa)$ holds.

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Likewise, $\mathrm{HL}^{tc}(d, \sigma, \kappa)$ is preserved by forcings of cardinality less than κ.

Reflection

Prop. (D/H) Suppose κ is either strongly inaccessible, or $\mathrm{cf}(\kappa) \geq \omega_1$ and κ is a limit of strongly inaccessibles. For $1 \leq d < \omega$ and $1 \leq \sigma < \kappa$, if $HL(d, \sigma, \alpha)$ holds for stationary many α, then $\mathrm{HL}(d, \sigma, \kappa)$ holds.

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Thm. (D/H) Suppose κ is measurable, $1 \leq d < \omega$, $1 \leq \sigma < \kappa$, and $\mathrm{HL}(d, \sigma, \kappa)$ holds. If $\mathbb{P}$ preserves stationary subsets of κ and adds no new bounded subsets of κ, then $\mathrm{HL}(d, \sigma, \kappa)$ holds after forcing with $\mathbb{P}$. In particular, $< \kappa$-closed forcings preserve $\mathrm{HL}(d, \sigma, \kappa)$.

Halpern-Läuchli at non weakly compact cardinals

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However, each of our method starts with a $\kappa + d$-strong cardinal.

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6. Are there any forcings which destroy HL(d, σ, κ)?
7. Does HL(d, σ, κ) simply hold for every strongly inaccessible κ?

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